m-th Roots of Matrices Revisited

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A matrix B is an m-th root of a matrix A if $A = B^m$ where $m > 1$ is an integer. Roots of matrices are important in accounting, cryptography, medical imaging, data analysis, etc.. For diagonal matrices and invertible matrices over algebraically closed fields m-th roots exists, however, this need not be true for singular matrices. The $d \times d$ nilpotent Jordan block is an example of a rootless matrix. A singular matrix can be written as a direct sum of an invertible and a nilpotent matrix up to similarity, hence, the existence of an m-th root is equivalent to the existence of the nilpotent part of the matrix. There are many articles which give the existence criteria in different ways, either using the consecutive ranks of the matrix, or the sizes of its Jordan blocks, or the multiplicities of its Jordan blocks [2]. We will present our version which is also in terms of the multiplicities of Jordan blocks [1]. In fact, our interest in the subject is originated by noticing a mistake in [2]. We observed that there is a nice pattern in the multiplicities of the Jordan form of a nilpotent matrix B and that of B^m. This allowed us to formulate a matrix M, such that $M\underline{\mathrm{b}}^T = \underline{\mathrm{a}}^T$ where $\underline{\mathrm{b}} = (b_1, \ldots, b_d)$ and $\underline{\mathrm{a}} = (a_1, \ldots, a_d)$ denotes the Jordan types of B and $A = B^m$, respectively, where the i-th coordinate is the number of $i \times i$ Jordan blocks in the Jordan form of the matrix. Thus, computation of the Jordan form of the m-th power of a nilpotent matrix is reduced to a single matrix multiplication; conversely, the existence of an m-th root of a nilpotent matrix is reduced to the existence of a nonnegative integer solution to the corresponding system of linear equations. We can also obtain new m-th roots from a given m-th root B, singular matrix A.
On the topological side there is a recent work [3] by Clement de Seguins Pazzis proving that the set of all m-th roots of a nilpotent complex matrix is path-connected. Computing the fundamental group (or higher homotopy groups) is an open question.

References

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