The q-Bernstein polynomials: why q?

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1 Introduction

The Bernstein polynomials first appeared in:

S. N. Bernstein, Démonstration du théorème de Weierstrass fondée sur la calcul des probabilités. *Communic. Soc. Math. Charkow*, série 2, 13, 1-2 (1912).

It is a very short paper which have made a huge impact on both mathematics and its application. The fact that the Bernstein polynomials are used, for example, in the CAGD, while there were no computers in 1912, can be attributed only to the intuition of a genius scientist.

In that paper, it was proved that, given $f \in C[0,1]$, polynomials

$$B_n(f;x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k(1-x)^{n-k},\ n = 1, 2, \ldots,$$

converge to f *uniformly* on $[0,1]$.

Certainly, this statement provides an elegant proof of the Weierstrass Approximation Theorem exhibiting approximation polynomials explicitly. Nowadays, these polynomials are called the *Bernstein polynomials* and it has been discovered that they possess many remarkable properties which have made them an area of intensive research and have stipulated their numerous applications in various disciplines. It is not surprising, therefore, that lots of generalizations and analogues of these polynomials have been proposed.

In this presentation, the generalizations related to the q-calculus will be discussed. First, let me expose some history.

Alexandru Lupaş (1942 - 2007) was the person who pioneered the work on the q-versions of the Bernstein polynomials. In 1987, he introduced a q-analogue of the Bernstein operator and investigated its approximation and shape-preserving properties.

A. Lupaş, A q-analogue of the Bernstein operator. *University of Cluj-Napoca, Seminar on numerical and statistical calculus.* Nr. 9, (1987)

Regrettably, the operators proposed by Lupaş remained unnoticed for a long while due to the very limited availability of his article published in regional conference proceedings.

Here is a scanned copy of this paper. I received it by mere chance from a Romanian colleagues after my talk on the q-Bernstein polynomials in 2002. I have noticed that, in some cases, the techniques used in the theory of the q-Bernstein polynomials can be applied to the Lupaş operators. My paper was published in 2006 by the "Rocky Mountain Journal of Mathematics" and it draw attention to the Lupaş's paper. Afterwards, deep theorems on these operators were obtained by O. Agratini, N. Mahmudov, V. S. Videnskii, H. Wang. Some applications to the CAGD have also appeared. I sent a reprint of my paper to A. Lupaş, as it occurred a few weeks before his unexpected death and, sadly, I do not know whether he saw it.

The most popular q-generalization of the Bernstein polynomials which appeared 10 years after Lupaş's paper and independently from Lupaş, belongs to G.M. Philips who constructed new polynomials known today as the *q-Bernstein polynomials*. This is what G. M. Phillips writes in regard to this matter: "The polynomials (7) (= the q-Bernstein polynomials) and rational functions (10) (= Lupaş q-analogue).... its properties." Anyway, the history cannot be reversed, and, at the moment, the q-Bernstein polynomials remain the most popular q-version of the Bernstein polynomials, attracting lots of interest and attention. Therefore, in this talk, the focus is on this q-version, although interesting and important results have been obtained for other q-operators.

In this presentation, I will overview some properties of the q-Bernstein polynomials and try to answer an inevitable question: what is a benefit of using q?

To present the subject, let us recollect the needed notations and definitions.
Let $q > 0$. For any $n = 0, 1, 2, \ldots$ the *q-integer* $[n]_q$ is defined by:

$$[n]_q := 1 + q + \cdots + q^{n-1} \ (n = 1, 2, \ldots), \ [0]_q := 0;$$

and the *q-factorial* $[n]_q!$ by:

$$[n]_q! := [1]_q[2]_q \ldots [n]_q \ (n = 1, 2, \ldots), \ [0]_q! := 1.$$

For integers $0 \leq k \leq n$, the *q-binomial* coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is defined by:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

Clearly, for $q = 1$,

$$[n]_1 = n, \ \ [n]_1! = n!, \ \ \begin{bmatrix} n \\ k \end{bmatrix}_1 = \binom{n}{k}.$$

We use the following standard notation:

$$(a;q)_0 := 1; \ \ (a;q)_n := \prod_{k=0}^{n-1}(1 - aq^k), \ \ n \in \mathbb{N}, \ \ (a;q)_\infty = \prod_{k=0}^{\infty}(1 - aq^k).$$

Notice that, for $q = 1$, $(x;q) = (1-x)^n$, while the infinite product is divergent whenever $q \geq 1$.

If $q \in (0,1)$ and $z \in \mathbb{C}$, then the infinite product $(z;q)_\infty = \prod_{j=0}^{\infty}(1 - q^j z)$ defines an entire function, which plays an important role in our considerations. Its Taylor expansion was first obtained by Euler:

$$(z;q)_\infty = \sum_{k=0}^{\infty} \frac{q^{k(k-1)/2}\,(-1)^k z^k}{(q;q)_k}; \tag{1.1}$$

as well as the expansion

$$\frac{1}{(z;q)_\infty} = \sum_{k=0}^{\infty} \frac{z^k}{(q;q)_k}, \quad |z|<1. \tag{1.2}$$

Definition 1. (G. M. Phillips) Let $f:[0,1]\to\mathbb{C},\ q>0$. The *$q$-Bernstein polynomial* of f is:

$$B_{n,q}(f;x) := \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right) p_{nk}(q;x), \quad n\in\mathbb{N}, \tag{1.3}$$

where

$$p_{nk}(q;x) = \begin{bmatrix} n \\ k \end{bmatrix}_q x^k (x;q)_{n-k}, \quad k=0,1,\dots,n. \tag{1.4}$$

It can be readily seen from the Definition that, when $q=1$, this polynomials coincide with the classical Bernstein polynomials. Conventionally, the name 'q-Bernstein polynomial' are reserved for the case $q\neq 1$.

2 How close are the q-Bernstein polynomials to the classical ones?

Naturally, it is desirable to know which properties of the classical Bernstein polynomials remain true for the q-analogues. This range of problems was investigated by their inventor and the related outcomes are summarized in:
G. M. Phillips, Interpolation and Approximation by Polynomials. Springer-Verlag, (2003)

To begin with, they possess the *end-point interpolation* property, that is

$$B_{n,q}(f;0)=f(0),\ B_{n,q}(f;1)=f(1) \ \text{ for all } \ q>0 \ \text{ and all } \ n\in\mathbb{N}.$$

Further, they leave the linear functions invariant:

$$B_{n,q}(ax+b)=ax+b \ \text{ for all } \ q>0 \ \text{ and all } \ n\in\mathbb{N}.$$

Also, the functions constituting the bases form partitions of 1; specifically:

$$\sum_{k=0}^{n} p_{nk}(q;x)=1, \quad x\in[0,1].$$

The theory of positive operators is also important in the case because, in the case $0<q<1$, the q-Bernstein polynomials generate *positive linear operators* on $C[0,1]$. What is more, they possess *shape-preserving* properties and admit probabilistic interpretation. Quite a few common features can be traced in the behaviour of the iterates.

The list of similarities with the classical case can be extended. For example, operators $B_{n,q}$ are degree-reducing on polynomials; that is, if T is a polynomial of degree m, then $B_{n,q}(T;x)$ is a polynomial of degree $\leq \min\{m,n\}$, and they are diagonalizable in the linear space of polynomials of degree $\leq n$ with an eigenstructure similar to that of the Bernstein polynomials. The estimates for the moments also show resemblance to the classical case. In the case $q = q_n \to 1^-$, the Korovkin Theorem holds and analogues of the Voronovskaya and Popoviciu theorems are valid.

However, this findings do not help to answer the question stated at the beginning: why do we need q? In this context, it is more interesting to look at the results, which have no counterparts in the classical case. In the sequel, the focus of this talk will be on the properties of the q-operators, which are different from those in the case $q = 1$.

3 Korovkin-type theorems

The Korovkin Theorem is one of the mostly used ones in the theory of approximation by positive operators.

Definition 2. An operator $T : C[0,1] \to C[0,1]$ is *positive* if

$$f(x) \geq 0, x \in [0,1] \Rightarrow (Tf)(x) \geq 0,\ x \in [0,1].$$

In the Approximation theory, the next theorem attributed to P. P. Korovkin is of a major importance.

Theorem 1. *(P. P. Korovkin) Let $\{T_n\}$ be a sequence of positive linear operators and $e_i(x) = x^i, i \in \mathbb{N}_0$. If*

$$(T_n e_i)(x) \to e_i(x) \text{ uniformly on } [0,1] \text{ for } i = 0,1,2,$$

then

$$(T_n f)(x) \to f(x) \text{ uniformly on } [0,1]$$

for every $f \in C[0,1]$. Functions $1,\ x,\ x^2$ are often referred to as test functions.

Although, for $0 < q < 1$, the q-Bernstein polynomials are positive linear operators on $C[0,1]$, they do not satisfy the conditions of the Korovkin theorem because

$$B_{n,q}(x^2) = x^2 + \frac{x(1-x)}{[n]_q} \to x^2 + (1-q)x(1-x) \neq x^2 \text{ for } x \in [0,1] \text{ as } n \to \infty. \tag{3.1}$$

Therefore, for $q \in (0,1)$ being fixed, the approximation of all continuous functions by the q-Bernstein polynomials operators is not possible. Nevertheless, the examination of their limit behavior results in the following general theorem by H. Wang.

Theorem 2. [9] *Let $\{L_n\}$ be a sequence of positive linear operators on $C[0,1]$ satisfying the following conditions:*

(a) *the sequence $\{L_n(x^2)\}$ converges uniformly on $[0,1]$;*

(b) *the sequence $\{L_n(f;x)\}$ is non-increasing in n for any convex function f and each $x \in [0,1]$.*

Then, there exists an operator L_∞ on $C[0,1]$ such that, for every $f \in C[0,1]$,

$$L_n(f;x) \to L_\infty(f;x) \;\; as \;\; n \to \infty \;\; uniformly \;\; on \;\; [0,1].$$

Remark 1. It should be emphasized that, in general, condition (b) cannot be left out. What is more, H. Wang proved by counterexample that even under some natural restrictions on operators L_n, the existence of the limit operator can fail.

This is an unusual situation to realize that the results on q-analogues of the classical operators triggered the discovery of a general result, while mostly the research on q-analogues follows the classical approaches.

It turns out that the q-Bernstein polynomials satisfy the conditions of Wang's theorem and, as such, possess a limit operator, that is, for $q \in (0,1)$, $\{B_{n,q}(f;x)\} \to f(x)$ uniformly on $[0,1]$. Meanwhile, the theorem on the limit of this sequence was proved independently of Wang's theorem and the emerging limit operator is denoted by $B_{\infty,q}$. Its properties will be discussed in subsequent sections. Here, we draw attention to the following theorem proposed by H. Wang, which is also applicable for the q-Bernstein polynomials.

Theorem 3. *(H. Wang) Let L be a positive linear operator on $C[0,1]$ which leaves invariant linear functions. If $L(t^2;x) > x^2$ for all $x \in (0,1)$, then $L(f) = f$ if and only if f is linear.*

As a result, in the case $q \in (0,1)$, the sequence $\{B_{n,q}(f;x)\}$ is not an approximating sequence for a function f unless f is linear. This by no means resembles the case $q = 1$, when $\{B_n(f;x)\}$ approximates f for any $f \in C[0,1]$. The lack of approximation did not annihilate the interest to study the q-analogues; yet, the focus was shifted to the properties of the limit operators and finding some new general facts. For example, the investigation of q-operators stimulated new researches on the convergence of the positive linear operators in the cases not covered by the Korovkin theorem.

4 On the limit operator

In the case $0 < q < 1$, the uniform convergence of $\{B_{n,q}(f;x)\}$ on $[0,1]$ is guaranteed by Wang's Theorem for each $f \in C[0,1]$. It is called the *limit q-Bernstein operator*. To present its explicit expression, let us consider the transcendental entire functions

$$p_{\infty k}(q;z) := \lim_{n\to\infty} p_{nk}(q;z) = \frac{z^k}{(q;q)_k}\,(z;q)_\infty,\; k = 0,1,\ldots$$

and set:

$$(B_{\infty,q}f)(x) = B_{\infty,q}(f;x) := \begin{cases} \sum_{k=0}^{\infty} f(1-q^k)p_{\infty k}(q;x), & \text{if } x \in [0,1), \\ f(1), & \text{if } x = 1. \end{cases}$$

Definition 3. Given $0 < q < 1$, the linear operator on $C[0,1]$ defined by:

$$B_{\infty,q} : f \mapsto B_{\infty,q} f$$

is called the *limit q-Bernstein operator*.

It has been proved that, for $0 < q < 1$, the sequence $\{B_{n,q}(f;x)\} \to B_{\infty,q} f(x)$ in $C[0,1]$. It is also a noteworthy fact that the same operator appears in the paper by Jing, where the q-deformed Poisson distribution is used to describe the energy distribution in a q-analogue of the coherent state and also in the paper of H. Wang as the limit for a sequence of the q-Meyer-König and Zeller operators. What is more, similar theorems have been proved for many various q-analogues, demonstrating, in this way, the similarity in the behaviour of such q-operators.

As for the properties of $B_{\infty,q}$, it can be stated that it is a bounded positive linear operator on $C[0,1]$ with $\|B_{\infty,q}\| = 1$. It possesses the end-point interpolation property and has the linear functions as their only fixed points. The kernel of this operators comprise functions

$$\{f \in C[0,1] : f(1-q^k) = 0,\ k \in \mathbb{N}_0\},$$

that is, functions vanishing on the time scale $\{1\} \cup \{1-q^k\}_{k=0}^{\infty}$. More developed investigation concerns with the analytical properties of $B_{\infty,q} f$.

5 The improvement of analytic properties under $B_{\infty,q}$.

It is not difficult to see that for $f \in C[0,1]$, the sequence $\left\{f(1-q^k)/(q;q)_k\right\}$ is bounded and, hence, $(B_{\infty,q} f)(x)$ admits an analytic continuation into the unit disc $\{z : |z| < 1\}$. Whenever $(B_{\infty,q} f)(x)$ admits an analytic continuation into a domain $D \subseteq \mathbb{C}$, we denote the continued function by $(B_{\infty,q} f)(z),\ z \in D \subseteq \mathbb{C}$. The following theorem shows that the possibility of an analytic continuation for $B_{\infty,q} f$ is affected by the smoothness of f at 1.

Theorem 4. *(S.O.)* *(i) If $f \in C[0,1]$ has m derivatives (from the left) at 1, then $B_{\infty,q} f$ admits analytic continuation into the disc $\{z : |z| < q^{-m}\}$. In particular, if f is infinitely differentiable (from the left) at 1, then $(B_{\infty,q} f)(z)$ is an entire function.*

(ii) Let $f \in C[0,1]$ have $m \geq 0$ derivatives at 1 and $f^{(m)}(x)$ satisfy the Lipschitz condition of order $\alpha\,(0 < \alpha \leq 1)$ at 1, that is for some $M > 0, |f^{(m)}(x) - f^{(m)}(1)| \leq M(1-x)^{\alpha}$. Then $(B_{\infty,q} f)(x)$ admits an analytic continuation into the disc $\{z : |z| < q^{-(m+\alpha)}\}$.

Remark 2. The results of Theorem 4 are sharp. In general, $B_{\infty,q} f$ may not admit an analytic continuation into a disc with radius greater than the one claimed by the theorem.

Theorem 4 states that if f is infinitely differentiable at 1, then $B_{\infty,q} f$ is entire. Further restrictions on f imply growth estimates for $B_{\infty,q} f$. Some exemplary results are presented by the next theorems.

For an entire function f, we denote

$$M(r; f) := \max_{|z| \leq r} |f(z)|\,.$$

Recall that, for $(z;q)_\infty = \prod_{j=0}^{\infty}(1-q^j z)$,

$$M(r;(z;q)_\infty) = (-r;q)_\infty \asymp \exp\left(\frac{\ln^2 r}{2\ln(1/q)} + \frac{\ln r}{2}\right).$$

Theorem 5. *(i) If $f \in C[0,1]$ has an analytic continuation into a closed disc $\{z : |z-1| \le q^a\}$, then*

$$M\left(r; B_{\infty,q}f\right) \le Cr^a(-r;q)_\infty, \quad r \ge 1.$$

(ii) If $f \in C[0,1]$ has an analytic continuation as an entire function, then

$$M(r; B_{\infty,q}f) \le Cr^{-u(r)}(-r;q)_\infty, \quad r \ge 1, \tag{5.1}$$

where $u(r) \to +\infty$ as $r \to \infty$.

Therefore, for any entire function f, its image $B_{\infty,q}f$ has a rather slow growth restricted by (5.1). This certainly implies that $B_{\infty,q}$ diminishes the growth of a rapidly growing entire function. Moreover, this appears to be a general phenomenon.

Theorem 6. *(S. O.) Let f be a transcendental entire function. Then $B_{\infty,q}f$ is also entire and*

$$M(r; B_{\infty,q}f) = o(M(r;f)), \quad r \to \infty.$$

Theorem 7. *(S. O.) If f is a polynomial, then $B_{\infty,q}f$ is also a polynomial with* $\deg f = \deg B_{\infty,q}f$. *Besides,*

$$B_{\infty,q}(1-x)^m = (1-x)(1-qx)\ldots(1-q^{m-1}x), \quad m \in \mathbb{N}_0. \tag{5.2}$$

The results above show how the analytic properties of f are transformed under $B_{\infty,q}$. If f at least satisfies the Lipschitz condition at 1, then, on the whole, it gets "better", unless f is a polynomial, that is "too good" to be improved. For functions without the Lipschitz condition, we need a certain regularity condition to guarantee the improvement on smoothness.

It turns out that, in some way, the analytic properties of f may be retrieved from those of $B_{\infty,q}f$, provided the following equivalence relation on $C[0,1]$ is considered:

$$f \sim g \Leftrightarrow B_{\infty,q}f = B_{\infty,q}g \Leftrightarrow f(1-q^k) = g(1-q^k), \ k \in \mathbb{N}_0.$$

The preceding reasonings lead to the following observation concerning the eigenvalues and eigenvectors of operator $B_{\infty,q}$: if f satisfies the Lipschitz condition (or any other condition stipulating an improvement of analytic properties) at 1 and also $B_{\infty,q}f(x) = \lambda f(x), \lambda \ne 0, x \in [0,1]$, then $\lambda = q^{m(m-1)/2}$, $m \in \mathbb{N}_0$ and f is a polynomial of degree m. The question whether the assertion remains true without any restrictions on f remains open, although it was posted at the MathOverflow site (http://mathoverflow.net/questions/223484/).

The result below describes the class of functions obtained as uniform limits of the q-Bernstein polynomials.

Theorem 8. *(S.O.) A function $g(x) = (B_{\infty,q}f)(x)$ for all $x \in [0,1]$ $\Leftrightarrow$ $f(x) = \sum_{k=0}^{\infty} a_k x^k$, where $\sum_{k=0}^{\infty} a_k$ converges. This implies that the image of $B_{\infty,q}$ does not depend on q.*

6 Probabilistic approach

As it has been mentioned, Bernstein introduced his polynomials using the considerations of Probability Theory. In particular, the basis Bernstein polynomials $\binom{n}{k} x^k(1-x)^{n-k}, \quad k = 0, 1, \ldots, n; \quad n \in \mathbb{N}$ express the probability of exactly k successes in the sequence of n Bernoulli trials with the probability x for a success in each trial.

A probabilistic interpretation of the q-Bernstein basis in the case $0 < q < 1$ was provided by A. Il'inskii and - in the case $q > 1$ - by Ch. A. Charalambides. See

Ch. A. Charalambides, q-discrete distributions.

Consider the following stochastic process. Let x and q be arbitrary numbers from the interval $(0, 1)$. Consider a sequence of trials $A_1, A_2, \ldots A_n$, where each trial has two outcomes, say, success (S) and failure (F). Suppose that $\mathbf{P}\{A_1 = S\} = x$, $\mathbf{P}\{A_1 = F\} = 1 - x$, and that for $k = 2, 3, \ldots, n$, the conditional probability $\mathbf{P}\{A_k = S|A_1, \ldots, A_{k-1}\} = q^j x$, where j is the number of failures among $A_1 \ldots A_{k-1}$. Correspondingly, $\mathbf{P}\{A_k = F|A_1, \ldots, A_{k-1}\} = 1 - q^j x$. We notice that In distinction from the classical case, trials $A_1, A_2, \ldots A_n$ are not independent and the probability for a success/failure depends on the history of the process. It is worth pointing out that in this scheme each failure diminishes the probability of a success in the next trial.

Theorem 9. *(A. Il'inskii) Denote by X_n a random variable giving the number of successes in n trials of the process described above. Then*

$$\mathbf{P}\{X_n = k\} = p_{nk}(q; x) = \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{j=0}^{n-1-j} (1 - q^j x), \; k = 0, 1, \ldots n.$$

Another connection with Probability Theory is related to a generalization of the Poisson distribution. Consider a function

$$\varphi(x) = \sum_{k=0}^{\infty} a_k x^k, \; a_0 = 1, \; a_k > 0$$

analytic in the disc $\{x : |x| < r\}$, $0 < r \leq \infty$ and consider a random variable ξ_x $(0 \leq x \leq r)$, whose values do not depend on x and are taken with the following probabilities:

$$\mathbf{P}\{\xi_x = \alpha_k\} = \frac{a_k x^k}{\varphi(x)} =: p_k(x), \; k \in \mathbb{N}_0.$$

Theorem 10. *(S.O.) Let ξ_x be a random variable with probability mass function above. Suppose that the following conditions are satisfied:*

(i) $\mathbf{E}[\xi_x] = x$;

(ii) $\mathbf{E}[\xi_x^2] = qx^2 + bx + c$.

Then, $c = 0, \; q > -1, \; \alpha_k = b[k]_q,$ *and* $a_k = \frac{1}{b^k [k]_q!}$.

Let X be the linear space of functions defined on $\{\alpha_k\}$ so that for $f \in X$ and $x \in [0, r)$, the mathematical expectation $\mathbf{E}\left[f(\xi_x)\right]$ exists. Then $(A_\varphi f)(x) := \mathbf{E}\left[f(\xi_x)\right]$ defines linear operator A_φ on X. Consider the following particular cases.

Example 1. Let $q = b = 1$. Then ξ_x has the Poisson distribution with parameter x. Correspondingly,

$$(A_\varphi f)(x) = \sum_{k=0}^{\infty} f(k) \frac{x^k}{k!} e^{-x}.$$

Example 2. For $q = 1$, $b = \frac{1}{n}$, one obtains:

$$(A_\varphi f)(x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) \frac{(nx)^k}{k!} e^{-nx} = S_n(f; x),$$

that is A_φ coincides with the Szász-Mirakyan operator.

Example 3. Let $0 < q < 1,\ b = 1 - q$. Then,

$$(A_\varphi f)(x) = (x; q)_\infty \sum_{k=0}^{\infty} \frac{f(1 - q^k)}{(q; q)_k} x^k = B_{\infty,q}(f; x).$$

Thence, in this way $B_{\infty,q}$ occurs as an analogue of the Szász-Mirakyan operator.

7 Convergence in the case $q > 1$

In the case $q > 1$, the similarity between the convergence properties of the q-Bernstein and the classical Bernstein polynomials does not persist any longer. Moreover, the convergence results for the q-Bernstein polynomials in the case $q > 1$ are not identical to those when $0 < q < 1$. To begin with, while, for $0 < q < 1$, operators $B_{n,q}$ are positive and $\|B_{n,q}\| = 1, n \in \mathbb{N}$, for $q > 1$, the positivity fails and the norms of $B_{n,q}$ increase in n rather rapidly, namely, $\|B_{n,q}\| \sim C_q q^{n(n-1)/2}/n,\ \ n \to \infty$ with $C_q = 2\,(q^{-2}; q^{-2})_\infty / e$. This immediately excludes the possibility of the q-Bernstein polynomials with $q > 1$ to be an approximating sequence of operators on $C[0, 1]$, although they are approximating on some everywhere dense subspaces. The next observation demonstrates that the timescale

$$\mathbb{J}_q := \{0\} \cup \{q^{-l}\}_{l=0}^{\infty}$$

plays a key role in this circle of problems.

Theorem 11. *(S. O., A. Y. Özban) The set $\mathbb{J}_q$ is a 'minimal' set, where the sequence $\{B_{n,q}(f; x)\}$ converges for all $f \in C[0, 1]$, in the sense that:*

(i) $\{B_{n,q}(f; x)\} \to f(x)$ when $x \in \mathbb{J}_q$, $f \in C[0, 1]$. Moreover, the convergence on $\mathbb{J}_q$ is uniform;

(ii) There exists $f \in C[0, 1]$, such that $q \geq 2,\ x \notin \mathbb{J}_q \Rightarrow |B_{n,q}(f; x)| \to \infty$ as $n \to \infty$.

It is easy to show by example that a sequence $\{B_{n,q}(f; x)\}$ may be divergent even for an infinitely differentiable functions:

Example 4. Let $q \in (1, \infty), a \in (1/q^2, 1/q)$. Consider a function $f \in C^\infty[0,1]$ satisfying

$$f(x) = \begin{cases} 0, & \text{for } 0 \le x \le 1/q^2, \\ 1, & \text{for } a \le x \le 1/q, \\ 0, & \text{for } x = 1. \end{cases}$$

In this case, $B_{n,q}(f; x) = [n]_q x^{n-1}(1-x) \to \infty$ for $x \in (1/q, 1)$, that is, the sequence of the q-Bernstein polynomials of f is divergent.

The convergence for the q-Bernstein polynomials in the case $q > 1$ has been studied in the number of papers under certain conditions regarding the possibility of an analytic continuation for f. The only presently available general result is the next saturation theorem:

Theorem 12. *(Z.X. Wu) The following assertion holds:*

$$\sup_{x\in[0,1]} |B_{n,q}(f; x) - f(x)| = o(q^{-n}) \Leftrightarrow f(x) = ax + b.$$

The statement ceases to be true if we replace 'o' with 'O'.

The results of the forthcoming section emphasize the connection between the convergence properties of the q-Bernstein polynomials and the analyticity of functions.

8 q-Bernstein polynomials and the analyticity of functions

It turns out that the properties of the q-Bernstein polynomials are strongly related to the possibility of a function to have an analytic continuation from $[0, 1]$ to a complex domain. Assume that f possesses an analytic continuation from $[0, 1]$ to a disc $D_R = \{z \in \mathbb{C} : |z| < R\}$ and consider the complex q-Bernstein polynomials $B_{n,q}(f; z)$. Let us start with the following theorem.

Theorem 13. *(S. O.) If $f \in C[0, 1]$ possesses an analytic continuation $f(z)$ into a disc D_R, then for any compact set $K \subset D_R$,*

$$B_{n,q}(f; z) \to f(z) \ \ uniformly\ \ on\ \ K\ \ as\ \ n \to \infty.$$

Corollary 14. *If $f \in C[0, 1]$ has an analytic continuation as an entire function $f(z)$, then, for any compact set $K \subset \mathbb{C}$, $B_{n,q}(f; z) \to f(z)$ uniformly on K as $n \to \infty$.*

The sharpness of Theorem 13 is to some extent provided by the next assertion:

Theorem 15. *(I. Ostrovskii, S. O.) Let $f(x) \in C[0, 1]$ admit an analytic continuation from $[0, 1]$ to D_R for some $R > 0$. If*

$$B_{n,q}(f; x) \to f(x)\ \ as\ \ n \to \infty$$

uniformly on $[0, 1]$, then $f(x)$ admits an analytic continuation into D_1.

Notice that the set of functions approximated by their q-Bernstein polynomials on $[0,1]$ in the case $q > 1$ is yet to be described.

Open problem. Is it possible that $B_{n,q}(f;x)$ converges on $[0,1]$ a function without an analytic continuation into any disc D_R?

Meanwhile, or functions admitting an analytic continuation, the following Voronovskaya - type theorems hold.

Theorem 16. *(H. Wang, X. Z. Wu) Given $R > q > 1$ and a function $f \in C[0,1]$ admitting an analytic continuation into the disc D_R. For $z \in D_{R/q}$, set:*

$$L_q(f;z) := \frac{(1-z)(D_q f(z) - f'(z))}{q-1},$$

where $D_q f$ denotes the q-derivative of f. Then

$$\lim_{n\to\infty} [n]_q (B_{n,q}(f;z) - f(z)) = L_q(f;z)$$

uniformly on any compact set $K \subset D_{R/q}$.

For the same class of functions, the following saturation of convergence for the q-Bernstein polynomials occurs.

Theorem 17. *(H. Wang, X. Z. Wu) Let $R > q > 1$. If a function f is analytic in the disc D_R, then $|B_{n,q}(f;z) - f(z)| = o(q^{-n})$ for an infinite number of points having an accumulation point in $D_{R/q}$ if and only if f is linear.*

Remark 3. In the same paper Wang and Wu proved the following saturation result for $q = 1$: if f is analytic in $D_R, R > 1$, then $|B_n(f;z) - f(z)| = o(1/n)$ for an infinite number of points having an accumulation point in the disc D_R if and only if f is linear. This result is new and gives an example of how the research on the q-Bernstein polynomials leads to new discoveries concerning the classical Bernstein polynomials!

9 q-Bernstein polynomials of discontinuous functions

Originally, the Bernstein polynomials were applied only for approximating the continuous functions. Later, L. V. Kantorovich first applied these polynomials to wider classes of functions. Specifically, he introduced the modified Bernstein polynomials, also regarded today as the *Kantorovich polynomials*, to approximate integrable functions on $[0,1]$ and, subsequently, G. G. Lorentz made use of these polynomials to approximate functions $f \in L^p[0,1]$ in the L_p-metric. Afterwards, Lorentz discussed approximation of unbounded functions by the Bernstein polynomials, stating "*Remarkable phenomena can occur for unbounded functions*". This has been reaffirmed by profound theorems proved by Chlodovsky, Herzog, Hill and Lorentz himself.
G. G. Lorentz, Bernstein Polynomials. Chelsea, New York, (1986).
In the same manner, initially, the q-Bernstein polynomials were considered solely for continuous functions. Nowadays, it can be seen that the investigation of the q-Bernstein polynomials

attached to discontinuous functions is also a beneficial aspect of research because, in this direction, new phenomena have been revealed. To be more specific, the behavior of these polynomials depends on whether singularity belongs to sequence $\mathbb{J}_q$, in which case the type of singularity impacts the approximation properties. Available information on this topic together with examples is given in:

S. Ostrovska, A. Y. Özban, M. Turan, How do singularities of functions affect the convergence of q-Bernstein polynomials? J. of Math. Inequal. **9** (1), 121-36 (2015)

We mention only the following illustrative results. Consider the set of functions $f : [0,1] \to \mathbb{R}$ continuous on $[0,1] \setminus \{\alpha\}$ and possessing an analytic continuation from $[0,\alpha)$ into $\mathcal{D}_\alpha$. The set of such functions will be denoted by $\mathcal{F}$.

Theorem 18. *(S. O., A. Y. Özban, M. Turan) Let $f \in \mathcal{F}$ and $\alpha \in (q^{-(m+1)}, q^{-m})$ for some $m \in \mathbb{N}_0$. Then f is uniformly approximated by its q-Bernstein polynomials on any compact set in $(-\alpha, \alpha)$.*

Comparing with Theorem 13 on the continuous functions admitting an analytic continuation into D_R, one can observe that the case $\alpha \notin \mathbb{J}_q$ cannot be viewed as a non-trivially new one. The situation changes if $\alpha \in \mathbb{J}_q$. According to the type of singularity, the following subsets of $\mathcal{F}$ are considered:

- $\mathcal{A} = \{f \in \mathcal{F} : \text{there exists } \gamma > 0 \text{ such that } \lim\limits_{x \to \alpha^-} f(x)(\alpha - x)^\gamma = K \in \mathbb{R} \setminus \{0\}\}$.
- $\mathcal{B} = \{f \in \mathcal{F} : \lim\limits_{x \to \alpha^-} f(x)(\alpha - x)^\gamma = \infty \text{ for all } \gamma > 0\}$.
- $\mathcal{C} = \{f \in \mathcal{F} : \lim\limits_{x \to \alpha^-} f(x)(\alpha - x)^\gamma = 0 \text{ for all } \gamma > 0\}$.

The next theorem shows that the sets of convergence for the q-Bernstein polynomials depend on whether $f \in \mathcal{A}, \mathcal{B}$, or $\mathcal{C}$.

Theorem 19. *(S. O., A. Y. Özban, M. Turan) (i) Let $f \in \mathcal{A}$. Then $f(x)$ is uniformly approximated by $B_{n,q}(f; x)$ on any compact set in $(-\alpha q^{-\gamma}, \alpha q^{-\gamma})$, while if $|x| > \alpha q^{-\gamma}$ and $x \notin \mathbb{J}_q$, then $|B_{n,q}(f; x)| \to \infty$ as $n \to \infty$.*

(ii) Let $f \in \mathcal{B}$. Then $|B_{n,q}(f; x)| \to \infty$ for $x \notin \{1, q^{-1}, \ldots, q^{-(m-1)}, 0\}$ as $n \to \infty$, and, obviously, $B_{n,q}(f; 0) = f(0)$.

(iii) Let $f \in \mathcal{C}$. Then $B_{n,q}(f; x) \to f(x)$ uniformly on any compact set in $(-\alpha, \alpha)$.

10 Variable q

From the very first papers, there was interest in approximation properties of q-Bernstein polynomials for q taking *varying* values that tend to 1. As a result, the case $0 < q < 1$, $q = q_n \uparrow 1$ has been studied in detail. The following theorem summarizes the obtained findings on the matter.

Theorem 20. *(V. S. Videnskii) If $0 < q_n < 1$, then for any $f \in C[0,1]$, the following estimate holds:*

$$|B_{n,q_n}(f;x) - f(x)| \leq 2\omega_f \left(\max\left\{1 - q_n, n^{-1}\right\}\right),$$

where ω_f is the modulus of continuity of f on $[0,1]$. Consequently, $B_{n,q_n}(f;x)$ converges uniformly to $f(x)$ on $[0,1]$ whenever $q_n \to 1^-$ as $n \to \infty$.

Naturally, the similar question arises when we deal with varying $q > 1$. Namely, given $f \in C[0,1]$ and $q_n \to 1^+$, is it true that $B_{n,q_n}(f;x) \to f(x)$ for $x \in [0,1]$ as $n \to \infty$? In general, the answer is 'no'. The example below suggests that the possibility of the approximation in $C[0,1]$ depends on the rate of $q_n \to 1$.

Example 5. If $B_{n,q_n}(\sqrt{x};x) \to \sqrt{x}$ uniformly on [0,1], then $q_n - 1 \leq C\frac{\ln n}{n}$.

A more subtle analysis leads to the following remarkable result by X.Z. Wu, which demonstrates that not only the order, but also a constant multiple in the behaviour of $q_n - 1$ affects the possibility of approximation for all continuous functions:

Theorem 21. *(X.Z. Wu) (i) Let $q_n > 1$. If, for every $f \in C[0,1]$, the sequence $\{B_{n,q_n}(f;x)\}$ converges to $f(x)$ uniformly on $[0,1]$, then*

$$\limsup_{n\to\infty} n(q_n - 1) \leq \ln 2.$$

(ii) Let $q_n > 1$. If the sequence $q_n \leq 1 + \frac{\ln 2}{n}$ for sufficiently large n, then for every $f \in C[0,1]$, the sequence $\{B_{n,q_n}(f;x)\}$ converges to $f(x)$ uniformly on $[0,1]$.

11 Conclusion

On the whole, the study of different q-analogues of the Bernstein polynomials has developed into a fruitful direction of research. Although many of the obtained results can be viewed as merely straightforward generalizations of those known, this study either requires different tools or leads to problems which cannot even arise in the classical setting, such as, for example, the dependence of outcomes on parameter q or the analytical and geometric properties of the limit operators. Their investigation establishes new connections between the relatively narrow class of operators and other areas, not only inside approximation theory, but also within functional analysis, algorithms, complex analysis, and others. The emergence of new phenomena adds further strength to this venture. Some of those results have been mentioned in this review, while others can be found in the cited references. Lately, new papers on applications of the q-analogues of the Bernstein polynomials have appeared. Nevertheless, open problems still exist in the theory and hopefully intriguing new results as well as interesting applications will be discovered in the future.

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