3.2 Application: Borsuk-Ulam Theorem

In this section we use cup products in order to prove the following result:

Theorem 3.2.1 (Borsuk-Ulam). *If $n > m \geq 1$, there are no maps $g : S^n \to S^m$ commuting with the antipodal maps, i.e., for which $g(-x) = -g(x)$, for all $x \in S^n$.*

Proof. We prove the theorem by contradiction. Assume that there is a map $g : S^n \to S^m$ commuting with the antipodal maps. Then g carries pairs of antipodal points $(x, -x)$ in S^n to pairs of antipodal points $\big(g(x), g(-x) = -g(x)\big)$ in S^m. So, by passage to the quotient, g induces a map

$$\begin{aligned} f : \mathbb{RP}^n &\to \mathbb{RP}^m \\ [x] &\mapsto [g(x)] \end{aligned}$$

which makes the following diagram commutative:

$$\begin{tikzcd} S^n \arrow[r, "g"] \arrow[d, "p'"'] \arrow[dr, phantom, "\circlearrowright"] & S^m \arrow[d, "p"] \\ \mathbb{RP}^n \arrow[r, dashed, "f"] & \mathbb{RP}^m \end{tikzcd}$$

Here p and p' are the two-sheeted covering maps.

We claim that there exists a lift f' of f, i.e., $f = pf'$ in the following diagram:

$$\begin{tikzcd} & S^m \arrow[d, "p"] \\ \mathbb{RP}^n \arrow[ur, dashed, "f'"] \arrow[r, "f"] & \mathbb{RP}^m \end{tikzcd}$$

Let us for now assume the claim and complete the proof of the theorem. Consider the following diagram:

$$\begin{tikzcd} & & S^m \arrow[d, "p"] \\ S^n \arrow[urr, "g"] \arrow[r, "p'"'] & \mathbb{RP}^n \arrow[ur, "f'"'] \arrow[r, "f"'] & \mathbb{RP}^m \end{tikzcd}$$

We have $pg = fp' = pf'p'$, the second equality following from the above claim. This implies that both g and $f'p'$ are lifts of fp'. Under the two-sheeted covering map p, antipodal points in S^m are mapped to the same point in $\mathbb{RP}^m$. Therefore, $pg = pf'p'$ implies that at a point $x \in S^n$, we have $g(x) = f'p'(x)$ or $ag(x) = f'p'(x)$, where $a : S^m \to S^m$ is the antipodal map. But $ag(x) = -g(x) = g(-x)$ and $f'p'(x) = f'p'(-x)$. Thus at $x \in S^n$, one of following equalities holds: $g(x) = f'p'(x)$ or $g(-x) = f'p'(-x)$. Since g and $f'p'$ are lifts of fp' and they coincide at a point, it follows by the uniqueness of the lift that $g = f'p'$. But this is a contradiction since $p'(x) = p'(-x)$, hence $f'p'(x) = f'p'(-x)$, while $g(x) \neq g(-x) = -g(x)$.

It remains to prove the claim. A lift for f exists iff

$$f_*(\pi_1(\mathbb{RP}^n)) \subseteq p_*(\pi_1(S^m)). \tag{3.2.1}$$

If $m = 1$, the only homomorphism

$$f_* : \pi_1(\mathbb{RP}^n) \cong \mathbb{Z}/2 \to \pi_1(\mathbb{RP}^1) \cong \mathbb{Z}$$

is the trivial one, so (3.2.1) is satisfied.
If $m > 1$, both groups $\pi_1(\mathbb{RP}^n)$ and $\pi_1(\mathbb{RP}^m)$ are $\mathbb{Z}/2$. We will use cup products to show that the induced map $f_* : \mathbb{Z}/2 \to \mathbb{Z}/2$ on fundamental groups is the trivial map. Let $\alpha_m \in H^*(\mathbb{RP}^m; \mathbb{Z}/2)$ and $\alpha_n \in H^*(\mathbb{RP}^n; \mathbb{Z}/2)$ be the generators of degree 1, and consider the induced ring homomorphism

$$f^* : H^*(\mathbb{RP}^m; \mathbb{Z}/2) \to H^*(\mathbb{RP}^n; \mathbb{Z}/2).$$

We have:

$$0 = f^*(\alpha_m^{m+1}) = f^*(\alpha_m)^{m+1},$$

so $f^*(\alpha_m) \in H^1(\mathbb{RP}^n; \mathbb{Z}/2)$ has order $m + 1 < n + 1$. Therefore,

$$f^*(\alpha_m) \neq \alpha_n.$$

Since $H^1(\mathbb{RP}^n; \mathbb{Z}/2) = \mathbb{Z}/2 = \langle \alpha_n \rangle$, this implies that

$$f^*(\alpha_m) = 0.$$

Let $i : \mathbb{RP}^1 \hookrightarrow \mathbb{RP}^n$ and $j : \mathbb{RP}^1 \hookrightarrow \mathbb{RP}^m$ be the inclusions obtained by setting all but the first two homogeneous coordinates equal to zero. By cellular cohomology, the map $j^* : H^1(\mathbb{RP}^m) \to H^1(\mathbb{RP}^1)$ is an isomorphism, so $j^*(\alpha_m)$ is the generator of $H^1(\mathbb{RP}^1)$, and in particular,

$$j^*(\alpha_m) \neq 0.$$

On the other hand,

$$(f \circ i)^*(\alpha_m) = i^*(f^*(\alpha_m)) = 0.$$

So $(f \circ i)^* \neq j^*$, hence the maps $f \circ i$ and j are not homotopic.

But the homotopy classes of i and j generate $\pi_1(\mathbb{RP}^n)$ and $\pi_1(\mathbb{RP}^m)$, respectively. So the homomorphisms

$$\begin{array}{rcl} f_* : \pi_1(\mathbb{RP}^n) \simeq \mathbb{Z}/2 & \longrightarrow & \pi_1(\mathbb{RP}^m) \simeq \mathbb{Z}/2 \\ {[i]} & \mapsto & [f \circ i] \neq [j] \end{array}$$

maps the generator $[i]$ to an element of $\mathbb{Z}/2$ other than the generator $[j]$, i.e., $f_* = 0$. This proves the claim, and completes the theorem. $\square$

MATH 538, Midterm Exam Solutions

Problem 1. Let $X = \mathbb{RP}^2 \vee S^2$

a) Write the cellular chain and cochain complex of X.

Soln) X has 1 0-cell $\langle v \rangle$ - 1 1-cell $\langle e \rangle$ & 2 2-cells $\langle \sigma, \tau \rangle$

$$0 \xrightarrow{0} C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{0} 0$$

$C_2(X) = \langle \sigma, \tau \rangle$, $C_1(X) = \langle e \rangle$, $C_0(X) = \langle v \rangle$

$$0 \xrightarrow{0} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\partial_2} \mathbb{Z} \xrightarrow{\partial_1} \mathbb{Z} \xrightarrow{0} 0 \qquad (*)$$

Since σ & τ attached to e by the identifications $\partial\sigma = 2e$ & $\partial\tau = 0$, then $\partial_2 = [2, 0]$ as matrix.

If $(1,0)$ & $(0,1)$ are basis elements for $\mathbb{Z}\oplus\mathbb{Z}$ then $[2,0]\begin{bmatrix}1\\0\end{bmatrix} = [2] = 2.[1]$ & $[2,0].\begin{bmatrix}0\\1\end{bmatrix} = [0]$.

$\partial_1 = 0$ since $\partial e = v - v = 0$.

Consider $(*)$ & apply $\mathrm{Hom}(-, \mathbb{Z})$ then we get

$$0 \xleftarrow{0} \mathrm{Hom}(\mathbb{Z}\oplus\mathbb{Z}, \mathbb{Z}) \xleftarrow{\partial_2^*} \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \xleftarrow{\partial_1^*} \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \xleftarrow{0} 0$$

$$0 \xleftarrow{0} \mathbb{Z} \oplus \mathbb{Z} \xleftarrow{\delta_2} \mathbb{Z} \xleftarrow{\delta_1} \mathbb{Z} \xleftarrow{0} 0$$

$\mathbb{Z}\oplus\mathbb{Z} = C^2(X)$, $\mathbb{Z} = C^1(X)$, $\mathbb{Z} = C^0(X)$

A basis for $C^0(X)$ is $\{\nu\}$ s.t. $\nu(v) = 1$ & a basis for $C^1(X)$ is $\{\alpha\}$ s.t. $\alpha(e) = 1$ & for $C^2(X)$ is $\{\lambda, \mu\}$ s.t. $\lambda(\sigma) = \mu(\tau) = 1$ & $\lambda(\tau) = \mu(\sigma) = 0$.

$\delta_1 = \partial_1^* = 0$ map $\qquad \delta_2 = (\partial_2)^* = \begin{bmatrix}2\\0\end{bmatrix}$

b) $\mathrm{Ker}\,\delta_2 = \{ m\alpha \mid \delta_2(m\alpha) = m\delta_2(\alpha) = 0\} \Leftrightarrow \{m(a\lambda + b\mu) = 0\} \Leftrightarrow \{m = 0\}$

or $\begin{bmatrix}2\\0\end{bmatrix}[m.1] = \begin{bmatrix}0\\0\end{bmatrix} \Leftrightarrow m = 0$.

$\mathrm{Im}\,\delta_2 = \{a\lambda + b\mu \mid \delta_2(\alpha) = a\lambda + b\mu\}$ or $\begin{bmatrix}2\\0\end{bmatrix}[m.1] = \begin{bmatrix}2m\\0\end{bmatrix} \cong 2\mathbb{Z}\oplus 0$

$H^0(X;\mathbb{Z}) = \ker\delta_1 / \mathrm{Im}\,\delta_0 = \mathbb{Z}/0 \cong \mathbb{Z}$.

$H^1(X;\mathbb{Z}) = \ker\delta_2/\mathrm{Im}\,\delta_1 = 0/0 = 0$.

$H^2(X,\mathbb{Z}) = \ker\delta_3/\mathrm{Im}\,\delta_2 = \mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}\oplus 0 \cong \mathbb{Z}_2 \oplus \mathbb{Z}$.

Problem 2. Let M be closed orientable mnf of dim $2n$. Show that if $H_{n-1}(M;\mathbb{Z})$ is torsion-free then $H_n(M;\mathbb{Z})$ is also torsion free.

Soln: Poincaré Duality: $H^k(M;\mathbb{Z}) \cong H_{n-k}(M;\mathbb{Z})$ U.C.Thm: $H^n(M;\mathbb{Z}) = \mathrm{Hom}(H_n(M),\mathbb{Z}) \oplus \mathrm{Ext}(H_{n-1}(M);\mathbb{Z})$

Let $k=n$ since M is of dim $2n$ then by P.D.

$$H^n(M;\mathbb{Z}) \cong H_{2n-n}(M;\mathbb{Z}) \cong H_n(M;\mathbb{Z})$$

By U.C.Thm, $H^n(M;\mathbb{Z}) = \mathrm{Hom}(H_n(M),\mathbb{Z}) \oplus \mathrm{Ext}(H_{n-1}(M);\mathbb{Z})$

But $H_{n-1}(M;\mathbb{Z})$ is torsion free so $\mathrm{Ext}(H_{n-1}(M);\mathbb{Z}) = 0$

So, $H^n(M;\mathbb{Z}) = \mathrm{Hom}(H_n(M),\mathbb{Z})$ is free. Hence $H_n(M;\mathbb{Z}) \underset{P.D}{\cong} H^n(M;\mathbb{Z})$ is torsion free.

Problem 3. Show that for any map $f: S^{2n} \to \mathbb{C}P^n$ the induced map in homology $f_*: H_{2n}(S^{2n};\mathbb{Z}) \to H_{2n}(\mathbb{C}P^n;\mathbb{Z})$ is trivial.

Soln: Consider the induced maps $f^*: H^n(\mathbb{C}P^n;\mathbb{Z}) \to H^n(S^{2n};\mathbb{Z})$

$$\begin{array}{ccc} H^n(\mathbb{C}P^n;\mathbb{Z}) \times H^n(\mathbb{C}P^n;\mathbb{Z}) & \xrightarrow{\cup} & H^{2n}(\mathbb{C}P^n;\mathbb{Z}) \\ \downarrow f^* \qquad \downarrow f^* & & \downarrow f^* \\ H^n(S^{2n};\mathbb{Z}) \times H^n(S^{2n};\mathbb{Z}) & \xrightarrow{\cup} & H^{2n}(S^{2n};\mathbb{Z}) \end{array}$$

By the naturality of cup product $f^*(\alpha\cup\alpha) = f^*(\alpha)\cup f^*(\alpha)$

but $H^n(S^{2n};\mathbb{Z}) = 0$ so $f^*(\alpha) = 0$ & $f^*(\alpha\cup\alpha) = 0 \Rightarrow f^* = 0$.

since $H_{2n-1}(\mathbb{C}P^n) = 0$ & $H_{2n-1}(S^{2n}) = 0$ then $H^{2n}(\mathbb{C}P^n) \cong \mathrm{Hom}(H_{2n}(\mathbb{C}P^n),\mathbb{Z})$

& $H^{2n}(S^{2n}) \cong \mathrm{Hom}(H_{2n}(S^{2n}),\mathbb{Z})$, i.e. f^* is the hom dual of f_*

$$\begin{array}{ccc} H_{2n}(S^{2n}) & \xrightarrow{f_*} & H_{2n}(\mathbb{C}P^n) \\ & \searrow^{f^*(\varphi)} & \downarrow \varphi \\ & & \mathbb{Z} \end{array}$$

$f^*(\varphi) = \varphi \circ f_*$ $\quad f^* = 0 \Rightarrow f_* = 0$.

$\therefore$ There is no map $f: S^{2n} \to \mathbb{C}P^n$ inducing nontrivial map on homology.

Problem 4. a) Describe the cohomology rings $H^*(S^1\times S^1;\mathbb{Z})$ & $H^*(\mathbb{C}P^n;\mathbb{Z})$ ③

Soln: $H^*(S^1\times S^1;\mathbb{Z}) \cong H^*(S^1;\mathbb{Z})\otimes H^*(S^1;\mathbb{Z})$

$H^*(S^1;\mathbb{Z}) = \{a.1+b\alpha \mid \alpha\in H^1(S^1;\mathbb{Z}),\ a,b\in\mathbb{Z}\}$ $\quad H^*(S^1)\otimes H^*(S^1) = \{a.1+b\alpha+c\beta+d\alpha\beta \mid |\alpha|=|\beta|=1\}$

Note that $\alpha\cup\alpha=0$ & $\beta\cup\beta=0$ but $\alpha\cup\beta\neq 0$.

~~Similarly~~ For $H^*(\mathbb{C}P^n;\mathbb{Z}) \cong \mathbb{Z}[\alpha]/\langle\alpha^{n+1}\rangle$ where $|\alpha|=2$.

since $\langle\alpha\rangle = H^2(\mathbb{C}P^n;\mathbb{Z})$ & $\langle\alpha^n\rangle = H^{2n}(\mathbb{C}P^n;\mathbb{Z})$

$\alpha^n\cup\alpha\in H^{2n+2}(\mathbb{C}P^n;\mathbb{Z})=0$ so $\alpha^{n+1}=0$.

i.e. $H^*(\mathbb{C}P^n;\mathbb{Z}) \cong \{a_0 1+a_1\alpha+a_2\alpha^2+\dots+a_n\alpha^n \mid |\alpha|=2\}$.

b) Show that if $n>m$, then any map $f:\mathbb{C}P^n\to\mathbb{C}P^m$ induce trivial homomorphism $f^*: H^2(\mathbb{C}P^m;\mathbb{Z})\to H^2(\mathbb{C}P^n;\mathbb{Z})$.

Soln: Let $f:\mathbb{C}P^n\to\mathbb{C}P^m$ inducing $f^*: H^2(\mathbb{C}P^m)\to H^2(\mathbb{C}P^n)$ nontrivial homomorphism with $m<n$, So we get a nontrivial homomorphism $H^*(\mathbb{C}P^m)\xrightarrow{f^*} H^*(\mathbb{C}P^n)$ where $\alpha\in H^2(\mathbb{C}P^m)$

$$\mathbb{Z}[\alpha]/\langle\alpha^{m+1}\rangle \xrightarrow{f^*} \mathbb{Z}[\beta]/\langle\beta^{n+1}\rangle$$ & $\beta\in H^2(\mathbb{C}P^n)$

since f^* is nontrivial ~~ring~~ homomorphism then $f^*(\alpha^{m+1}) = f^*(0)=0$,

but $f^*(\alpha^{m+1}) = (f^*(\alpha))^{m+1} = (k\beta)^{m+1} = k^{m+1}\beta^{m+1}\neq 0$ since $m<n$.

$\therefore$ There is no ~~nontrivial~~ map inducing non-trivial homomorphism on H^2's. ∎

Problem 5: Show that a compact mnf. does not retract onto its boundary.

Soln: Assume that M retracts onto its boundary ∂M so $\exists\ r: M\to\partial M$ s.t. for $i:\partial M\hookrightarrow M$ $r\circ i = id_{\partial M}$

Since i is inclusion it induces injection on homology

$i_*: H_{n-1}(\partial M;\mathbb{Z}_2)\to H_{n-1}(M;\mathbb{Z}_2)$

$\longrightarrow$

Consider l.e.s. for pairs

$$\to H_n(\partial M; \cancel{\mathbb{Z}_2}) \to H_n(M; \cancel{\mathbb{Z}_2}) \to H_n(M, \partial M; \cancel{\mathbb{Z}_2}) \xrightarrow{\partial} H_{n-1}(\partial M; \mathbb{Z}_2) \xrightarrow[1-1]{i_*} H_{n-1}(M; \mathbb{Z}_2)$$

$H_n(\partial M) = 0$

By exactness $\ker i_* = \operatorname{Im} \partial \Rightarrow \partial = 0$.

But this can not happen since by Poincaré–Lefschetz Duality $\exists \in H_n(M, \partial M)$ which maps onto $[\partial M] \in H_n(\partial M)$ by ∂-map.

So ∂ can not be 0. $\lightning$ $\therefore$ There is no retraction $r: M \to \partial M$. $\blacksquare$

Problem 6: a) Show that for any compact oriented mnfld.

$$\chi(M) \equiv \sigma(M) \pmod 2$$

Soln: . If n is odd then $\chi(M) = 0$ & $\sigma(M) = 0$

. If $n \equiv 2 \pmod 4$ then $\chi(M)$ is even & $\sigma(M) = 0$.

. If $n = 4k$, then $\sigma(M) = r - s$ where $r + s = \dim H^{2k}(M; \mathbb{R}) \equiv \chi(M) \pmod 2$

$r + s \equiv r - s \pmod 2$.

b) Compute the signature of $\mathbb{C}P^2$; $\sigma(\mathbb{C}P^2)$

Soln: $H^2(\mathbb{C}P^2; \mathbb{R}) \cong \mathbb{R} = \langle \alpha \rangle$

$\langle \alpha \cup \alpha \rangle = H^4(\mathbb{C}P^2) \Rightarrow \alpha^2 \neq 0$ $\quad$ α^2 is dual of $[\mathbb{C}P^2]$

$\langle \alpha \cup \alpha, [\mathbb{C}P^2] \rangle = 1$

$\Phi: H^2(\mathbb{C}P^2) \times H^2(\mathbb{C}P^2) \longrightarrow \mathbb{R}$

$\Phi = [1] \Rightarrow \sigma(\mathbb{C}P^2) = 1 - 0 = 1$.

c) Does $\mathbb{C}P^2$ bound a 5-mnfld? Explain.

Soln: No, By Thm of Rokhlin-Thom if $M = \partial W$ then $\sigma(M) = 0$. $\sigma(\mathbb{C}P^2) = 1 \neq 0$ so $\mathbb{C}P^2$ does not bound a 5-mnfld.

$\blacksquare$ ✓

(Hatcher 3.1. #8)

Question 2: Show that if $f: S^n \to S^n$ has degree d then $f^*: H^n(S^n; G) \longrightarrow H^n(S^n; G)$ is multiplication by d.

Solution:

$$f_*: H_n(S^n; \mathbb{Z}) \longrightarrow H_n(S^n; \mathbb{Z})$$
$$x \longmapsto dx$$

$$H^n(S^n; G) \simeq \underbrace{\mathrm{Ext}(H_{n-1}(S^n); G)}_{=0} \oplus \mathrm{Hom}(H_n(S^n); G) = \mathrm{Hom}(H_n(S^n); G)$$

So we get a isomorphism $h: H^n(S^n; G) \longrightarrow \mathrm{Hom}(H_n(S^n), G) = \mathrm{Hom}(\mathbb{Z}, G)$

$$\begin{tikzcd}
H^n(S^n; G) \arrow[r, "h", "\simeq"'] \arrow[d, "f^*"'] & \mathrm{Hom}(\mathbb{Z}, G) \arrow[d, "?\ \circ f_*"] & \alpha \arrow[d, mapsto] \\
H^n(S^n; G) \arrow[r, "h", "\simeq"'] & \mathrm{Hom}(\mathbb{Z}, G) & \alpha \circ f_*
\end{tikzcd}$$

Consider $f_*: H_n(S^n) \longrightarrow H_n(S^n)$ let $\alpha \in \mathrm{Hom}(H_n(S^n), G)$
$$x \longmapsto dx$$

$$\begin{tikzcd}
H_n(S^n) \arrow[r, "f_*"] \arrow[dr, dashed, "\alpha \circ f_*"'] & H_n(S^n) \arrow[d, "\alpha"] \\
& G
\end{tikzcd}$$

Observe $\alpha \in \mathrm{Hom}(\mathbb{Z}, G)$ i.e. $\alpha: \mathbb{Z} \to G$, so we can identify it by $\alpha(1) \in G$. Then $(\alpha \circ f_*)(1) = \alpha(d.1) = d.\alpha(1)$

So, for any element $x \in H^n(S^n; G)$ say, $h(x) = \alpha$ then

$$(h^{-1} \circ (\circ f_*) \circ h)(x) = h^{-1}(\underbrace{\alpha \circ f_*}_{d\alpha}) \overset{\text{or}}{=} h^{-1}(h(x) \circ f_*) = h^{-1}(d\, h(x))$$
$$= d(h^{-1} h(x))$$
$$= dx.$$

Hence, $f^*(x) = dx$

$\blacksquare$

Hatcher 3.2 #3 : a) Using the cup product structure, show there is no map $\mathbb{R}P^n \to \mathbb{R}P^m$ inducing a nontrivial map $H^1(\mathbb{R}P^m;\mathbb{Z}_2) \to H^1(\mathbb{R}P^n;\mathbb{Z}_2)$ if $n>m$. What is the corresponding result for $\mathbb{C}P^n \to \mathbb{C}P^m$?

Solution: Let $f: \mathbb{R}P^n \to \mathbb{R}P^m$ be a map inducing a nontrivial map on $H^1(\mathbb{R}P^m;\mathbb{Z}_2) \xrightarrow{f^*} H^1(\mathbb{R}P^n;\mathbb{Z}_2)$, for $m<n$.

Then we get a nontrivial ~~map~~ homomorphism

$$\begin{array}{ccc} H^*(\mathbb{R}P^m;\mathbb{Z}_2) & \xrightarrow{f^*} & H^*(\mathbb{R}P^n;\mathbb{Z}_2) \\ \| & & \| \\ \mathbb{Z}_2[\alpha]/\langle\alpha^{m+1}\rangle & \xrightarrow{f^*} & \mathbb{Z}_2[\beta]/\langle\beta^{n+1}\rangle \end{array}$$

where $\alpha \in H^1(\mathbb{R}P^m;\mathbb{Z}_2)$ & $\beta \in H^1(\mathbb{R}P^n;\mathbb{Z}_2)$

Consider

$$f^*(\alpha^{m+1}) = f^*(0) = 0 \quad \text{(ring homomorphism)}$$

over $\mathbb{Z}_2$!

on the other hand $f^*(\alpha^{m+1}) = (f^*(\alpha))^{m+1} = (\beta)^{m+1} \neq 0$ since $m<n$

so $0 = f^*(\alpha^{m+1}) = (f^*(\alpha))^{m+1} \neq 0$ ↯ which is contradiction.

Hence there is no map $\mathbb{R}P^n \to \mathbb{R}P^m$ inducing nontrivial map on H^1's with $m<n$.

The corresponding result for $\mathbb{C}P^n \to \mathbb{C}P^m$: There is no map $\mathbb{C}P^n \to \mathbb{C}P^m$ inducing a nontrivial map on H^2's with $m<n$.

If so; $f: \mathbb{C}P^n \to \mathbb{C}P^m$ then

$$H^2(\mathbb{C}P^m;\mathbb{Z}) \xrightarrow{f^*} H^2(\mathbb{C}P^n;\mathbb{Z}) \quad \text{for } m<n$$

& we get

$$\begin{array}{ccc} H^*(\mathbb{C}P^m;\mathbb{Z}) & \xrightarrow{f^*} & H^*(\mathbb{C}P^n;\mathbb{Z}) \\ \| & & \| \\ \mathbb{Z}[\alpha]/\langle\alpha^{m+1}\rangle & \xrightarrow{f^*} & \mathbb{Z}[\beta]/\langle\beta^{n+1}\rangle \end{array}$$

$\alpha \in H^2(\mathbb{C}P^m;\mathbb{Z})$
$\beta \in H^2(\mathbb{C}P^n;\mathbb{Z})$

$$0 = f^*(\alpha^{m+1}) = (f^*(\alpha))^{m+1} = (k\beta)^{m+1} = k^{m+1}\cdot\beta^{m+1} \neq 0$$

since $m<n$.

↯

#8 b) Prove the Borsuk-Ulam Thm. by using part a.

Soln: By using the hint we'll prove the Borsuk-Ulam Thm: For every map $f: S^n \to \mathbb{R}^n$, there exists a point $x \in S^n$ with $f(x) = f(-x)$.

Suppose the contrary, that $f: S^n \to \mathbb{R}^n$ satisfies $f(x) \neq f(-x)$ $\forall x \in S^n$.

Define $g: S^n \to S^{n-1}$ by $g(x) = \dfrac{f(x) - f(-x)}{|f(x) - f(-x)|}$ $\quad g(x) = -g(-x)$ $\forall x \in S^n$

and

$$\begin{tikzcd}
\{x,-x\} \arrow[d, mapsto] \arrow[rrr, mapsto, bend left] & S^n \arrow[r, "g"] \arrow[d, "q_n"] & S^{n-1} \arrow[d, "q_{n-1}"] & \{g(x), -g(x)\} \arrow[d, mapsto] \\
{[x]} \arrow[rrr, mapsto, bend right] & \mathbb{R}P^n \arrow[r, "\bar{g}"'] \arrow[ur, dashed, "\tilde{g}"] & \mathbb{R}P^{n-1} & {[g(x)]}
\end{tikzcd}$$

So g defines a function $\bar{g}$ on the quotient.

Now, by part a) $(\bar{g})^*: H^1(\mathbb{R}P^{n-1}; \mathbb{Z}_2) \to H^1(\mathbb{R}P^n; \mathbb{Z}_2)$ must be zero map since $n-1 < n$. By U.C.T. $H^1(\mathbb{R}P^n; \mathbb{Z}_2) \cong \mathrm{Hom}(H_1(\mathbb{R}P^n), \mathbb{Z}_2) \oplus \underbrace{\mathrm{Ext}(H_0(\mathbb{R}P^n), \mathbb{Z}_2)}_{= 0}$

So,

$$\begin{tikzcd}
H_1(\mathbb{R}P^n) \arrow[rr, "(\bar{g})_*"] \arrow[dr, "\alpha"'] & & H_1(\mathbb{R}P^{n-1}) \arrow[dl, "\beta"] \\
& \mathbb{Z}_2 &
\end{tikzcd}$$

$(\bar{g})^*$ is zero map so $(\bar{g})^*(\beta) = 0$ $\forall \beta \in H^1(\mathbb{R}P^{n-1}; \mathbb{Z}_2)$

but $0 = (\bar{g})^*(\beta) = (\bar{g})_* \circ \beta \Rightarrow (\bar{g})_* = 0$ map.

Since $H_1(\mathbb{R}P^n) = \pi_1(\mathbb{R}P^n)$ then the map induced on π_1's must also be zero. So, by the lifting criteria $\exists\, \tilde{g}: \mathbb{R}P^n \to S^{n-1}$ which lifts $\bar{g}$. Now, consider the following diagram

$$\begin{tikzcd}
& & & S^{n-1} \arrow[d, "q_{n-1}"] \\
& & \mathbb{R}P^n \arrow[ur, "\tilde{g}"] \arrow[d, "\tilde{g}"'] \arrow[r, "\bar{g}"] & \mathbb{R}P^{n-1} \\
S^n \arrow[urrr, bend left, "g"] \arrow[urr, "q_n"] \arrow[rr, "g"'] & & S^{n-1} \arrow[ur, "q_{n-1}"'] &
\end{tikzcd}$$

~~By~~ From the diagram both g and $\tilde{g}\, q_n$ are lifts of $q_{n-1} g$. By the lifting theorem if two lifts of same map agree at a point then they must be same.

$\tilde{g}([x]) = \{g(x), -g(x)\}$ if g & $\tilde{g} q_n$ do not agree then g & $(-\tilde{g}) q_n$ agrees.

$\tilde{g}\, q_n(x) = \tilde{g}([x]) = \tilde{g}\, q_n(-x) \Rightarrow (\tilde{g} q_n)(x) = (\tilde{g} q_n)(-x) \Rightarrow \tilde{g} q_n$ is even by $g(x) = -g(-x)$ i.e. g is odd. So they can't be same. $\lightning$

Hence $f(x) = f(-x)$ for at least one $x \in S^n$. $\boxtimes$

Hatcher 3.2 #4: Show that every map $f: \mathbb{C}P^n \to \mathbb{C}P^n$ has a fixed point if n is even. When n is odd show $\exists$ a fixed point unless $f^*(\alpha) = -\alpha$ for $\alpha \in H^2(\mathbb{C}P^n;\mathbb{Z})$.

Soln: Leftschetz Fixed Point thm: If $\tau(f) \neq 0$ for $f: X \to X$ then f has a fixed point!

$$\tau(f) = \sum_k (-1)^k \, tr\left(f_* : H_k(\mathbb{C}P^n) \to H_k(\mathbb{C}P^n)\right)$$

By the naturality, for f

$$\begin{tikzcd}
0 \arrow[r] & Ext(H_{k-1}(\mathbb{C}P^n), \mathbb{Z}) \arrow[r] \arrow[dr, phantom, "\circlearrowleft"] & H^k(\mathbb{C}P^n;\mathbb{Z}) \arrow[r] \arrow[dr, phantom, "\circlearrowleft"] & Hom(H_k(\mathbb{C}P^n), \mathbb{Z}) \arrow[r] & 0 \\
0 \arrow[r] & Ext(H_{k-1}(\mathbb{C}P^n), \mathbb{Z}) \arrow[r] \arrow[u, "(f_*)^*_E"] & H^k(\mathbb{C}P^n;\mathbb{Z}) \arrow[r] \arrow[u, "f^*"] & Hom(H_k(\mathbb{C}P^n), \mathbb{Z}) \arrow[r] \arrow[u, "(f_*)^*"] & 0
\end{tikzcd}$$

Also, recall $H_k(\mathbb{C}P^n;\mathbb{Z}) = \begin{cases} \mathbb{Z} & k \text{ even}, \ 2 \le k \le 2n \\ 0 & o/w \end{cases} \Rightarrow H^k(\mathbb{C}P^n;\mathbb{Z}) = \begin{cases} \mathbb{Z} & k \text{ even}, \ 0 \le k \le 2n \\ 0 & o/w. \end{cases}$

and trace of a map being additive $tr(f^*) = tr(f_*)^*_E + tr(f_*)^*$

When k is odd $H_k(\mathbb{C}P^n) = 0$ & $H_{k-1}(\mathbb{C}P^n) = \mathbb{Z}$ but $Ext(\mathbb{Z},\mathbb{Z}) = 0$.

when k is even $H_k(\mathbb{C}P^n) = \mathbb{Z}$ & $H_{k-1}(\mathbb{C}P^n) = 0$ again $Ext(0,\mathbb{Z}) = 0$

So $tr(f^*) = tr(f_*)^* = tr(f_*)$.

Hence, $\tau(f) = \sum_k (-1)^k \, tr\left(f^* : H^k(\mathbb{C}P^n;\mathbb{Z}) \to H^k(\mathbb{C}P^n;\mathbb{Z})\right)$

Recall, $H^*(\mathbb{C}P^n;\mathbb{Z}) = \mathbb{Z}[\alpha]/\langle \alpha^{n+1} \rangle$, $|\alpha| = 2$. Then the ring homomorphism $f^*: H^*(\mathbb{C}P^n;\mathbb{Z}) \to H^*(\mathbb{C}P^n;\mathbb{Z})$ induces homomorphisms $f^*: H^k(\mathbb{C}P^n;\mathbb{Z}) \to H^k(\mathbb{C}P^n;\mathbb{Z})$ $\forall k$. In particular for $f^*: H^0(\mathbb{C}P^n;\mathbb{Z}) \to H^0(\mathbb{C}P^n;\mathbb{Z})$ is the identity $tr(f^*) = 1$ and $f^*: H^2(\mathbb{C}P^n;\mathbb{Z}) \to H^2(\mathbb{C}P^n;\mathbb{Z})$ By the naturality of the cup product!

$$\begin{tikzcd}
H^2(\mathbb{C}P^n;\mathbb{Z}) \arrow[rr, "f^*"] \arrow[ddd, "\cup"'] & & H^2(\mathbb{C}P^n;\mathbb{Z}) \arrow[ddd, "\cup"] \\
\alpha \arrow[r, mapsto] \arrow[d, mapsto] & m\alpha \arrow[d, mapsto] & \\
\alpha^i \arrow[r, mapsto] & m^i\alpha^i & \\
H^{2i}(\mathbb{C}P^n;\mathbb{Z}) \arrow[rr, "f^*"] & & H^{2i}(\mathbb{C}P^n;\mathbb{Z})
\end{tikzcd}$$

Note that f^* is a $\mathbb{Z}$-module homomorphism!

#4: $\tau(f) = \sum_{i=0}^{n} (-1)^{2i} m^i$ since $tr(f^*) = \underset{(m^i)}{\cancel{m}^i}$ $\forall i$.

$= \sum_{i=0}^{n} m^i = \begin{cases} n+1 & \text{if } m=1 \\ \frac{m^{n+1}-1}{m-1} & \text{if } m \neq 1 \end{cases}$ In any case as long as $m\neq 1$ and n is odd this number is nonzero.

Therefore, if n is even $\tau(f) \neq 0$ so f must have a fixed point, but for n is odd if $f^*(\alpha) = -\alpha$ then one can explicitly write a map $f: \mathbb{C}P^n \longrightarrow \mathbb{C}P^n$ which is given by

$$[a_1 : a_2 : a_3 : \dots : a_n : a_{n+1}] \longmapsto [-\bar{a}_2 : \bar{a}_1 : -\bar{a}_4 : \bar{a}_3 : \dots : -\bar{a}_{n+1} : \bar{a}_n]$$

(where $n+1$ must be even $\Rightarrow$ n must be odd)

This map has no fixed points: If $[a_1 : a_2 : \dots : a_n : a_{n+1}] = [b_1 : b_2 : \dots : b_n : b_{n+1}]$ then $a_i = \lambda b_i$ ~~&~~ $a_{i+1} = \lambda b_{i+1}$ and so on. So $\frac{a_i}{b_i} = \frac{a_{i+1}}{b_{i+1}}$ or $a_i b_{i+1} = a_{i+1} \cdot b_i$

if $[a_1 : a_2 : \dots : a_n : a_{n+1}] = [-\bar{a}_2 : \bar{a}_1 : \dots : -\bar{a}_{n+1} : \bar{a}_n] \Rightarrow a_1 \bar{a}_1 = a_2 \cdot (-\bar{a}_2)$

in general $a_i \bar{a}_i = -(a_{i+1} \cdot \bar{a}_{i+1}) \Rightarrow |a_i|^2 = -|a_{i+1}|^2 \Leftrightarrow a_i = a_{i+1} = 0$.

Hence f has no fixed points.

<u>Hatcher 3.2. #11</u>: Show that every map $S^{k+l} \to S^k \times S^l$ induces trivial homomorphism $H_{k+l}(S^{k+l}) \longrightarrow H_{k+l}(S^k \times S^l)$ for $k, l > 0$.

<u>Solution:</u> Recall $H_i(S^k \times S^l) = \begin{cases} \mathbb{Z} & \text{for } i = k, l, k+l \\ 0 & \text{o/w}. \end{cases}$

By U.C.T. $H^i(S^k \times S^l; \mathbb{Z}) \cong \text{Hom}(H_i(S^k \times S^l), \mathbb{Z}) \oplus \underbrace{\text{Ext}(H_{i-1}(S^k \times S^l), \mathbb{Z})}_{=0}$

$= \begin{cases} \mathbb{Z} & \text{for } i = k, l, k+l, 0 \\ 0 & \text{o/w}. \end{cases}$

But $H^i(S^{k+l}; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } i = 0, k+l \\ 0 & \text{o/w} \end{cases}$

So $f^*: H^i(S^k \times S^l; \mathbb{Z}) \longrightarrow H^i(S^{k+l}; \mathbb{Z})$ induces zero homomorphism for $i = k$ & l. On the other hand by the Künneth formula

$H^*(S^k \times S^l; \mathbb{Z}) \cong H^*(S^k; \mathbb{Z}) \otimes H^*(S^l; \mathbb{Z})$. Any element in $H^{k+l}(S^k \times S^l; \mathbb{Z})$ can be written as a product of elements in $H^{k+l}(S^k \times S^l; \mathbb{Z})$ & $H^{k+l}(S^k \times S^l; \mathbb{Z})$ i.e.

$$H^{\cancel{k+l}}(S^k \times S^l; \mathbb{Z}) \times H^{\cancel{k+l}}(S^k \times S^l; \mathbb{Z}) \xrightarrow{\cup} H^{k+l}(S^k \times S^l; \mathbb{Z})$$

#11 -

By the naturality of the cup product

$$\begin{array}{ccccc} H^k(S^k\times S^l;\mathbb{Z}) & \times & H^l(S^k\times S^l;\mathbb{Z}) & \xrightarrow{\cup} & H^{k+l}(S^k\times S^l;\mathbb{Z}) \\ \downarrow f^*=0 & & \downarrow f^*=0 & & \downarrow f^* \\ H^k(S^{k+l};\mathbb{Z}) & \times & H^l(S^{k+l};\mathbb{Z}) & \xrightarrow{\cup} & H^{k+l}(S^{k+l};\mathbb{Z}) \end{array}$$

so $f^*(\alpha\cup\beta) = f^*(\alpha)\cup f^*(\beta)$ hence this f^* must be zero.

Now, consider U.C.T again, because it is natural

$$\begin{array}{ccccccccc} 0 \to & \mathrm{Ext}(H_{k+l-1}(S^k\times S^l),\mathbb{Z}) & \longrightarrow & H^{k+l}(S^k\times S^l;\mathbb{Z}) & \xrightarrow{h} & \mathrm{Hom}(H_{k+l}(S^k\times S^l),\mathbb{Z}) & \to 0 \\ & \downarrow (f_*)^* & \circlearrowright & \downarrow f^* & \textcircled{1}\circlearrowright & \downarrow (f_*)^* & \\ 0 \to & \mathrm{Ext}(H_{k+l-1}(S^{k+l}),\mathbb{Z}) & \longrightarrow & H^{k+l}(S^{k+l};\mathbb{Z}) & \xrightarrow{h} & \mathrm{Hom}(H_{k+l}(S^{k+l});\mathbb{Z}) & \to 0 \end{array}$$

By the commutativity of ① $(f_*)^*h = hf^* = 0$ map since f^* is zero & h is onto, if $(f_*)^*h = 0$ then $(f_*)^* = 0 \Rightarrow f_* = 0$

therefore, $f_* : H_{k+l}(S^{k+l}) \longrightarrow H_{k+l}(S^k\times S^l)$ is zero map.

Math 538 HMW 3 Solns:

Hatcher 3.3 #3: Show that every covering space of an orientable manifold is an orientable manifold.

Solution: Let $p: \tilde{M} \to M$ be a covering map. Since M is orientable choose an orientation i.e. a generator for $H_n(M, M-\{x\})$ for all $x \in M$. Let $B_x \cong D^n$ be a nghd of x. Since p is a covering map then B_x is evenly covered by $\tilde{B}_{x_i}$'s where $p^{-1}(x) = \{x_i\}$ and for each i $p: \tilde{B}_{x_i} \to B_x$ is a (local) homeomorphism.

So the generator $\mu_{B_x} \in H_n(M, M-B_x)$ projects to μ_y $\forall y \in B_x$

$$\begin{tikzcd} & \mu_{B_x} \in H_n(M, M-B_x) \arrow[dl] \arrow[d] \arrow[dr] & \\ \mu_y & H_n(M, M-\{y\}) & H_n(M, M-\{x\}) \ni \mu_x \end{tikzcd}$$

Let $x_i \in \tilde{M}$, take a ball $B_{p(x_i)}$ around $p(x_i) \in M$ and $\tilde{B}_{x_i}$ be the ball around x_i where $p: \tilde{B}_{x_i} \to B_{p(x_i)}$ is a (local) homeomorphism.

then $H_n(\tilde{M}, \tilde{M}-\{x_i\}) \underset{\text{excision}}{\cong} H_n(\tilde{B}_{x_i}, \tilde{B}_{x_i}-\{x_i\}) \underset{\text{local homeo}}{\cong} H_n(B_{p(x_i)}, B_{p(x_i)}-\{p(x_i)\})$

$\underset{\text{excision}}{\cong} H_n(M, M-\{p(x_i)\})$

So, we may choose some generator for $H_n(\tilde{M}, \tilde{M}-\{x_i\})$

Let $y_i \in \tilde{B}_{x_i}$, then consider $\mu_{\tilde{B}_{x_i}} \in H_n(\tilde{M}, \tilde{M}-\tilde{B}_{x_i}) \cong H_n(\tilde{M}, \tilde{M}-\{x_i\})$

$\cong H_n(M, M-\{p(x_i)\})$

$\cong H_n(M, M-B_{p(x_i)})$

Since $p: \tilde{B}_{x_i} \to B_{p(x_i)}$ is a local homeo. then $p(y_i) = y$ for some $y \in B_{p(x_i)}$ & $\mu_{B_{p(x_i)}} \longmapsto \mu_y$

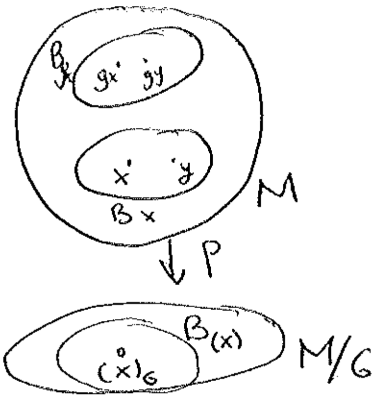
so $\mu_{\tilde{B}_{x_i}} \longmapsto \mu_{y_i}$ by local compatibility condition.

Hatcher 3.3. #4: Given a covering space action of G on an orientable manifold M by orientation preserving homeomorphisms, show that M/G is also orientable.

Soln: G is a covering space action $\Rightarrow$ for any $x \in M$ and any ~~open~~ ball B_x around x $B_x \cap gB_x = \emptyset$, ~~And~~ $g: M \to M$ is an orientation preserving homeomorphism means $g_*: H_n(M, M-\{x\}) \longrightarrow H_n(M, M-\{gx\})$

$$\mu_x \longmapsto \mu_{gx}$$

For any $y \in B_x$ $\mu_{B_x} \longmapsto \mu_y$, $\mu_{B_x} \longmapsto \mu_x$ ~~by~~ local consistency holds since M is orientable.



(x) = orbit of $x = \{gx \mid g \in G\}$

~~#~~ $p: B_x \longrightarrow B_{(x)}$ is a homeomorphism so

$$H_n(B_x, B_x - \{x\}) \cong H_n(B_{(x)}, B_{(x)} - \{(x)\})$$

$\cong$ excision $\qquad$ $\cong$ excision

$$H_n(M, M-\{x\}) \cong H_n(M/G, M/G - \{(x)\})$$

$M/G = \{(x) \mid (x) \text{ orbits of } x \ \forall x \in M\}$

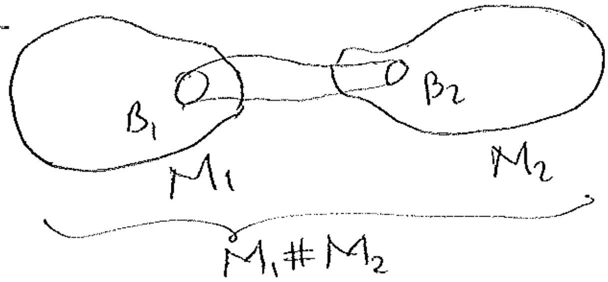
So, $\mu_x \longmapsto \mu_{(x)}$ well-defined since $\mu_x \mapsto \mu_{gx}$ $\forall g \in G$.

Finally, local consistency holds in $B_{(x)}$ since it holds in B_x. Therefore M/G is also orientable.

Hatcher 3.3. #6: Given two disjoint connected n-mnflds M_1 & M_2 consider $M_1 \# M_2$ connected sum.

a) Show that if M_1 & M_2 are closed then $H_i(M_1 \# M_2; \mathbb{Z}) \approx H_i(M_1; \mathbb{Z}) \oplus H_i(M_2; \mathbb{Z})$ with one exception if both M_1 & M_2 are non-orientable, then $H_{n-1}(M_1 \# M_2; \mathbb{Z})$ is obtained by replacing one of the two $\mathbb{Z}_2$ summands by a $\mathbb{Z}$-summand.

Soln:



$B_i = D^n$ $\quad \partial B_i = S^{n-1}$

Now consider the pair $(M_1 \# M_2, S^{n-1})$ which is a good pair so we have

Relative homology l.e.s.

$$0 \to H_n(S^{n-1}) \longrightarrow H_n(M_1 \# M_2) \to H_n(M_1 \# M_2, S^{n-1}) \to H_{n-1}(S^{n-1}) \to \cdots$$

#6: $M_1 \# M_2 / S^{n-1} \simeq$  $= M_1 \vee M_2$

So we have

$0 \to \underbrace{H_n(S^{n-1})}_{=0} \to H_n(M_1 \# M_2) \to H_n(M_1 \vee M_2) \to H_{n-1}(S^{n-1}) \to H_{n-1}(M_1 \# M_2)$

$\Rightarrow 0 \to H_n(M_1 \# M_2) \to H_n(M_1) \oplus H_n(M_2) \to \mathbb{Z} \to H_{n-1}(M_1 \# M_2) \to H_{n-1}(M_1) \oplus H_{n-1}(M_2) \to$

If M_1 & M_2 are orientable then $H_n(M_i) = \mathbb{Z}$ $\forall i = 1, 2$.

so, $0 \to H_n(M_1 \# M_2) \to \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\varphi} \mathbb{Z} \to H_{n-1}(M_1 \# M_2) \to H_{n-1}(M_1) \oplus H_{n-1}(M_2)$

$\Rightarrow \ker \varphi = H_n(M_1 \# M_2) = \begin{cases} \mathbb{Z} \\ \mathbb{Z} \oplus \mathbb{Z} \end{cases}$ $M_1 \# M_2$ is closed & connected & φ is onto?

So there is one n-dim'l cell so $H_n(M_1 \# M_2) = \mathbb{Z} \Rightarrow M_1 \# M_2$ is orientable.

If $M_1 \# M_2$ is orientable then we have

$0 \to H_n(M_1 \# M_2) \to H_n(M_1) \oplus H_n(M_2) \xrightarrow{\varphi} H_{n-1}(S^{n-1}) \to \cdots$

$0 \to \mathbb{Z} \underset{\text{injective}}{\xrightarrow{i}} H_n(M_1) \oplus H_n(M_2) \underset{\text{onto}(?)}{\xrightarrow{\varphi}} \mathbb{Z} \to \cdots$

 $M_1 - B_1 / S^{n-1} \simeq M_1$

So $H_n(M_1) \oplus H_n(M_2) = \mathbb{Z} \oplus \mathbb{Z}$ then M_1 & M_2 are orientable.

Now, for $i < n-1$ $\tilde{H}_i(S^{n-1}) = 0 \Rightarrow \tilde{H}_i(M_1 \# M_2) = \tilde{H}_i(M_1) \oplus \tilde{H}_i(M_2)$

for $i = n-1$ If one of M_1 & M_2 are orientable then

$H_n(M_1) = \mathbb{Z}$ ~~&~~ $M - B_1 / S^{n-1} \simeq M_2 \Rightarrow H_n(M_1) = \mathbb{Z}$ & $H_{n-1}(S^{n-1}) = \mathbb{Z}$

, the map φ must be onto which gives $H_{n-1}(M_1 \# M_2) = H_{n-1}(M_1) \oplus H_{n-1}(M_2)$

If both M_1 & M_2 are non-orientable then $M_1 \# M_2$ is non-orientable

$H_n(M_1 \# M_2) = 0 = H_n(M_1) = H_n(M_2)$ so we get

$0 \to \underset{\substack{\| \\ \mathbb{Z}}}{H_{n-1}(S^{n-1})} \to H_{n-1}(M_1 \# M_2) \to H_{n-1}(M_1) \oplus H_{n-1}(M_2) \to \underset{\substack{\| \\ 0}}{H_{n-2}(S^{n-1})}$

Now, by corollary 3.28 $H_{n-1}(M_1 \# M_2)$ has torsion $\mathbb{Z}/2$ since it is non-orientable. Similarly $H_{n-1}(M_1)$ & $H_{n-1}(M_2)$ has torsion $\mathbb{Z}_2$

#6: So we have

$$0 \to \mathbb{Z} \xrightarrow{\varphi} F \oplus \mathbb{Z}_2 \to F_1 \oplus F_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \to 0 \quad \text{exact}$$

$$\Rightarrow \quad 0 \to \mathbb{Z} \xrightarrow{\varphi} F \oplus \mathbb{Z}_2 \to F \oplus \mathbb{Z}_2 / \mathrm{Im}\,\varphi \to 0$$

$F \oplus \mathbb{Z}_2 / \mathrm{Im}\,\varphi = F_1 \oplus F_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ so $\mathbb{Z} \xhookrightarrow{\times 2} F$ must hold! Hence we get the desired result.

b) Show that $\chi(M_1 \# M_2) = \chi(M_1) + \chi(M_2) - \chi(S^n)$.

$$\begin{aligned}
\chi(M_1 \# M_2) &= \sum_{i=0}^{n} (-1)^n \operatorname{rk}(H_i(M_1 \# M_2)) \\
&= (-1)^0 \operatorname{rk}(H_0(M_1 \# M_2)) + \sum_{i=1}^{n-2} (-1)^i \operatorname{rk}(H_i(M_1 \# M_2)) \\
&\quad + (-1)^{n-1} \operatorname{rk}(H_{n-1}(M_1 \# M_2)) + (-1)^n \operatorname{rk}(H_n(M_1 \# M_2)) \\
&= 1 + \sum_{i=1}^{n-2} (-1)^i \operatorname{rk}(H_i(M_1) \oplus H_i(M_2)) + (-1)^{n-1} \operatorname{rk}(H_{n-1}(M_1 \# M_2)) + (-1)^n \operatorname{rk}(H_n(M_1 \# M_2)) \\
&= 1 + \sum_{i=1}^{n-2} (-1)^i \operatorname{rk}(H_i(M_1)) + \sum_{i=1}^{n-2} (-1)^i \operatorname{rk}(H_i(M_2)) + (-1)^{n-1} \operatorname{rk}(H_{n-1}(M_1 \# M_2)) + (-1)^n \operatorname{rk}(H_n(M_1 \# M_2))
\end{aligned}$$

$$= \begin{cases}
\displaystyle\sum_{i=0}^{n} (-1)^i \operatorname{rk}(H_i(M_1)) + \sum_{i=0}^{n} (-1)^i \operatorname{rk}(H_i(M_2)) \underbrace{- 1 + (-1)^{n+1}} & \text{if both } M_1 \,\&\, M_2 \text{ are orientable} \\
\text{same} & \text{if one of them is orientable the other is non-orientable} \\
\displaystyle\sum_{i=0}^{n-1} (-1)^i \operatorname{rk}(H_i(M_1)) + \sum_{i=0}^{n-1} (-1)^i \operatorname{rk}(H_i(M_2)) - 1 + (-1)^{n-1} & \text{if none of them is orientable}
\end{cases}$$

$$(-1) + (-1)^{n+1} = (-1) + (-1)^{n-1} = -[1 + (-1)^n] = -\chi(S^n)$$

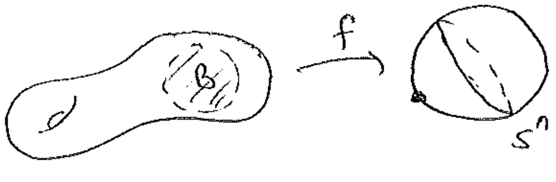
$$\therefore \quad \chi(M_1 \# M_2) = \chi(M_1) + \chi(M_2) - \chi(S^n).$$

MATH 538 HMW 4 Solutions

Hatcher 3.3.#7: Show that for any connected, closed, orientable n-mnfld. M there is a degree 1 map $M \to S^n$.

Solution: Let $[M]$ be the fundamental class of M, which is exists since M is orientable closed n-mnfld. Let $B \subset M$ be an open disk. Define a map $f: M \to S^n$ as follows

send B to $S^n - \{pt\}$ $\quad M - B$ to $\{pt\}$



Actually $M/M{-}B \xrightarrow{g} S^n$ is a homeomorphism.

So, on homology this gives us

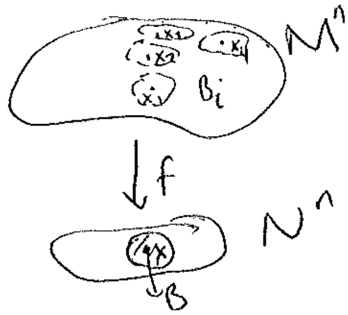
$$\begin{array}{ccc} H_n(M) & \xrightarrow{\cong} & H_n(M, M-B) \\ {[M]} & \longmapsto & \mu_B \\ f_* \downarrow & & \Vert \, \cong \\ H_n(S^n) & \xrightarrow[\cong]{g_*} & H_n(M/M{-}B) \end{array}$$

Since ~~have~~ have these isomorphism then f_* has to be isomorphism

So $f_*([M]) = \pm [S^n]$. If it is $-[S^n]$ then we can choose the other orientation of M, $[-M]$.

3.3.#8: $f: M^n \to N^n$, connected, closed, orientable. Suppose $\exists$ a ball $B \subset N$ s.t. $f^{-1}(B) = \coprod_i B_i$ s.t. $f(B_i) \underset{\text{homeo}}{\cong} B$. Show that degree of f is $\sum_i \varepsilon_i$ where $\varepsilon_i = \pm 1$

Soln:



M orientable then

$$H_n(M) \longrightarrow H_n(\cancel{M}, M - B_i) \cong H_n(M, M - \{x_i\})$$
$$[M] \longmapsto \mu_{x_i}$$
$$H_n(M) \xrightarrow{\cong} H_n(B_i, B_i - \{x_i\}) \cong H_n(M, M-\{x_i\})$$

$$H_n(B_i, B_i - \{x_i\}) \cong H_n(M, M - \{x_i\}) \xrightarrow{i} H_n(M, M - \{x_1, \dots, x_k\})$$

Also, we have $H_n(N) \longrightarrow H_n(N, N - \cancel{B}) \cong H_n(N, N - \{x\}) \cong H_n(B, B - \{x\})$
$$[N] \longmapsto \mu_x$$

$$\begin{tikzcd}
H_n(B_i, B_i-\{x_i\}) \arrow[r, "(f|_{B_i})_*", "\cong"'] \arrow[d, "\text{inc}\ k_i"] & H_n(B, B-\{x\}) \arrow[d, "\cong"] \\
H_n(M, M-\{x_1,\dots,x_n\}) \arrow[r, "f_*"] & H_n(N, N-\{x\}) \\
H_n(M) \arrow[r, "f_*"] \arrow[u, "J"] & H_n(N) \arrow[u, "\cong"]
\end{tikzcd}$$

commutes

Consider also,

$$\begin{tikzcd}
 & H_n(M, M-\{x_1,\dots,x_n\}) \cong \oplus \mathbb{Z} \arrow[d, "p_i"] & \\
H_n(B_i, B_i-\{x_i\}) \arrow[ur, "\text{inc}\ k_i"] \arrow[r, "\cong"'] & H_n(M, M-\{x_i\}) & H_n(M) \arrow[ul, "J"'] \arrow[l, "\cong"]
\end{tikzcd}$$

commutes

$p_i J([M]) = \mu_{x_i}$ & $p_i(k_i(\mu_{x_i})) = \mu_{x_i} \Rightarrow J([M]) = \sum k_i(\mu_{x_i})$

so $(f|_{B_i})_*(\mu_{x_i}) = \mu_x$ so $f_*(k_i(\mu_{x_i})) = \pm 1 = \varepsilon_i \pm[\mu_x]$

so $f_*([M]) = d[N]$ $f_*(J([M])) = f_*\left(\sum k_i(\mu_{x_i})\right) = \sum f_* k_i(\mu_{x_i})$

so $f_*([M]) = \left(\sum f_* k_i(\mu_{x_i})\right)[N] \Rightarrow d = \left(\sum f_* k_i(\mu_{x_i})\right)$.

3.3 # 8: M, N connected, closed, orientable n-mflds. Show that a p-sheeted covering space projection $M \xrightarrow{\pi} N$ has degree $\pm p$.

Soln: Apply the previous question to π.

$$\pi_*([M]) = d[N] \quad \text{where} \quad d = \sum_{i=1}^{p} \pi_* k_i(\mu_{x_i}) \quad \text{since by}$$

3.3. # 3. $\pi_* k_i(\mu_{x_i}) = +1$ or -1 then $d = \pm p$.

3.3.#17: Show that direct limit of exact sequences is exact. More generally, homology commutes with direct limits.

~~Solution~~ If $\{C_\alpha, f_{\alpha\beta}\}$ is a directed system of chain complexes with $f_{\alpha\beta}: C_\alpha \to C_\beta$ chain maps, then $H_n(\varinjlim C_\alpha) = \varinjlim H_n(C_\alpha)$.

Solution: Let $0 \to A_i \xrightarrow{g_\alpha} B_i \xrightarrow{f_\alpha} C_i \to 0$ be exact

3.3. #17

and $\varinjlim A_i = A \quad \varinjlim B_i = B \quad \varinjlim C_i = C$ (3)

show $0 \to A \xrightarrow{g} B \xrightarrow{f} C \to 0$ exact.

Now, let $x \in B$ s.t. $f(x) = 0$ or $x \in \ker f$ then $f_\alpha(x) = 0 \in C_\alpha$. for B_α containing x & since $0 \to A_\alpha \xrightarrow{g_\alpha} B_\alpha \xrightarrow{f_\alpha} C_\alpha \to 0$ is exact then $\exists\, a \in A_\alpha$ s.t. $g_\alpha(a) = x$ so $g(a) = x$, ie. $x \in \operatorname{Im} g$

$\Rightarrow \ker f \subseteq \operatorname{Im} g$. If $x \in \operatorname{Im} g$, $\exists\, a \in A$ s.t. $g(a) = x$, $a \in A_\alpha$ for some α so $x = g_\alpha(a)$ $f_\alpha g_\alpha(a) = f_\alpha(x) = 0$ then $f(x) = 0$. $\Rightarrow \operatorname{Im} g \subseteq \ker f$.

For homology: Consider

$$0 \to (Z_n)_\alpha \longrightarrow (C_n)_\alpha \xrightarrow{\partial_n} (C_{n-1})_\alpha \quad \text{which is exact}$$

then by above $0 \to \varinjlim (Z_n)_\alpha \longrightarrow C_n \xrightarrow{\partial_n} C_{n-1}$ is exact

$$\ker \partial_n = \varinjlim (Z_n)_\alpha = Z_n$$

Similarly,
$$0 \to (Z_{n+1})_\alpha \to (C_{n+1})_\alpha \xrightarrow{\partial_{n+1}} (B_n)_\alpha \to 0$$

$$0 \to Z_{n+1} \to C_{n+1} \xrightarrow{\partial_{n+1}} \varinjlim (B_n)_\alpha \to 0 \qquad \operatorname{Im} \partial_{n+1} = \varinjlim (B_n)_\alpha = B_n$$

Also $0 \to (B_n)_\alpha \longrightarrow (Z_n)_\alpha \longrightarrow (H_n)_\alpha \to 0$ we have

$$0 \to \underbrace{\varinjlim (B_n)_\alpha}_{\shortparallel\, B_n} \to \underbrace{\varinjlim (Z_n)_\alpha}_{\shortparallel\, Z_n} \longrightarrow \varinjlim (H_n)_\alpha \to 0 \quad \text{is exact.}$$

$$\Rightarrow \varinjlim (H_n)_\alpha = H_n(\varinjlim C_n).$$