3.3. Poincaré Duality: (manifolds, locally homeomorphic to $\mathbb{R}^n$)

~~Orientation~~ **Defn:** A manifold of dimension n, n-manifold is a Hausdorff space M in which each point has a open nghd homeomorphic to $\mathbb{R}^n$. The dim of M characterized by local homology groups

$$0 \neq H_i(M, M-\{x\}; \mathbb{Z}) \approx H_i(\mathbb{R}, \mathbb{R}^n - \{0\}; \mathbb{Z}) \quad \text{by excision}$$

only for i

$$\approx \widehat{H}_{i-1}(\mathbb{R}^n - \{0\}; \mathbb{Z}) \quad \text{since } \mathbb{R}^n \text{ is contractible}$$

$$\approx \widehat{H}_{i-1}(S^{n-1}; \mathbb{Z}) \quad \text{since } \mathbb{R}^n - \{0\} \simeq S^{n-1}$$

Defn A compact mnf without boundary is called "closed"

eg. S^n, $\mathbb{R}P^n$, $\mathbb{C}P^n$, $L(p,q)$, M, N closed $\Rightarrow$ $M \times N$ closed.

Poincaré duality: • A closed, orientable, n-manifolds ~~of dim n~~ M, there are isomorphisms $H_k(M; \mathbb{Z}) \approx H^{n-k}(M; \mathbb{Z})$ $\forall k$. $(0 \leq k \leq n)$

• Without orientability we have $H_k(M; \mathbb{Z}_2) \approx H^{n-k}(M; \mathbb{Z}_2)$

• Also, since the homology groups of a closed mnf are finitely generated

$$H_k(M; \mathbb{Z}) \approx H_{n-k}(M; \mathbb{Z}) \quad \text{modulo torsion subgroups and}$$

$\text{torsion}(H_k(M; \mathbb{Z})) \approx \text{torsion}(H_{n-k-1}(M; \mathbb{Z}))$ for torsion subgroups.

Poincaré duality expresses a certain symmetry in the homology of a closed orientable mnfd.

e.g. T^n, by induction on n it follows from Künneth formula

$$H_k(T^n; \mathbb{Z}) \approx \bigoplus_{\binom{n}{k}} \mathbb{Z} \qquad \binom{n}{k} = \binom{n}{n-k}$$

$$\approx \bigoplus_{\binom{n}{n-k}} \mathbb{Z} \approx H_{n-k}(T^n; \mathbb{Z})$$

There is a nice geometric proof of Poincaré duality by the notion of dual cell structures. Tetrahedron$^{\circlearrowleft *}$, cube $*$ octahedron, dodecahedron $*$ icosahedron. Given a pair of dual cell structures C & C^* on a closed surface M; $C_0^* = C_2$, $C_1^* = C_1$, $C_2^* = C_0$. If we use $\mathbb{Z}$ coefficients then there is an ambiguity of sign for each cell, the choice of generator for the corresponding $\mathbb{Z}$ summand of the cellular chain cx. If we use $\mathbb{Z}_2$-coeff then $C_i = C_{2-i}^*$ is canonical

$\partial : C_i \longrightarrow C_{i-1}$ becomes $\delta : C^*_{2-i} \longrightarrow C_{2-i+1}$

since ∂ assigns to a cell the sum of the cells which are faces of it, while δ assigns to a cell the sum of the cells of which it is a face. Thus, $H_i(C; \mathbb{Z}_2) \simeq H^{2-i}(C^*; \mathbb{Z}_2)$

For, $\mathbb{Z}$-coeffnents, if M is orientable, it is possible to make consistent choices of orientations of C & C^* so that the ∂ maps agree.

For manifolds of higher dimension the situation is analogous.

Each i-cell of C being dual to $(n-i)$-cell of C^* which it intersects in one point "transversely"

Orientations & Homology:

What is the orientation of $\mathbb{R}^n$? Preserved under rotations & reversed under reflections.

An orientation of $\mathbb{R}^n$ at x is a choice of the generator $H_n(\mathbb{R}^n, \mathbb{R}^n - \{x\}) \simeq H_{n-1}(\mathbb{R}^n - \{x\}) \simeq H_{n-1}(S^{n-1})$ ~~where S^{n-1}~~

Rotations of S^{n-1} have degree 1, being homotopic to identity while reflections have degree -1. An orientation at y

$H_n(\mathbb{R}^n, \mathbb{R}^n - \{x\}) \simeq H_n(\mathbb{R}^n, \mathbb{R}^n - B) \simeq H_n(\mathbb{R}^n, \mathbb{R}^n - \{y\})$ ~~where~~

where B contains x & y. This can be extended to mnflds.

Defn: A "local orientation" of M at a point x is a choice of a generator μ_x of the infinite cyclic group $H_n(M, M - \{x\})$.

An "orientation" of M is a function $x \longmapsto \mu_x$ ($M \longmapsto \coprod_{x \in M} H_n(M, M-\{x\})$) assigning to each $x \in M$ a local orientation $\mu_x \in H_n(M, M - \{x\})$ satisfying local consistency condition s.that each $x \in M$ has a nghd $\mathbb{R}^n \subset M$ containing open ball B of finite radius about x s.t. all the local orientations μ_y at $y \in B$ are images of of one generator μ_B of $H_n(M, M-B) \simeq H_n(\mathbb{R}^n, \mathbb{R}^n - B)$ under the natural map $H_n(M, M-B) \simeq H_n(M, M - \{y\})$

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If an orientation exist then M is called orientable.

<u>Prop</u>: Every mfd M has an orientable two-sheeted covering space $\tilde{M}$.

<u>Proof</u>: M be an n-mnfd. Let $\tilde{M} = \{\mu_x \mid x \in M \text{ \& } \mu_x \text{ is a local orientation}\}$

$$p: \tilde{M} \longrightarrow M$$
$$\mu_x \longmapsto x$$ is a 2-1 surjection. Topologize $\tilde{M}$ to make p a covering map, as follows: Given $B \subset \mathbb{R}^n \subset M$ of finite radius and a generator $\mu_B \in H_n(M, M-B)$, define $U(\mu_B) = \{\mu_x \mid x \in B$ & $\mu_B \mapsto \mu_x$ under $H_n(M, M-B) \to H_n(M, M-\{x\})$. Then

$\{U(\mu_B) \mid B \subseteq \mathbb{R}^n \subset M\}$ is a basis for the topology on $\tilde{M}$.

$\tilde{M}$ is orientable, for $\mu_x \in \tilde{M}$ has a canonical orientation

$$\tilde{\mu}_x \in H_n(\tilde{M}, \tilde{M}-\mu_x) \approx H_n(U(\mu_B), U(\mu_B)-\mu_x) \approx H_n(B, B-\{x\})$$
$$\tilde{\mu}_x \longmapsto \mu_x$$

<u>Prop 3.25.</u> If M is connected, then M is orientable iff $\tilde{M}$ has two components. In particular, if M is simply-connected or more generally, if $\pi_1(M)$ has no subgroup of index 2 then M is orientable.

<u>Proof</u>: If M is connected, $\tilde{M}$ has either one or two components. If it has two components, they're each homeomorphically mapped onto M, so M is orientable being homeomorphic to a component of an orientable mnfd. $\tilde{M}$. Conversely, if M is orientable it has two orientations & each of these orientations defines a component of $\tilde{M}$. The last statement; connected two-sheeted covering spaces of M corresponds to index-two subgroup of $\pi_1(M)$.

<u>Rmk</u>: For a ~~ring~~ commutative ring R with identity an R-orientation on M^n is a function
$$M \longrightarrow \coprod_{x\in M} H_n(M, M-\{x\}; R)$$
$$x \longmapsto \mu_x$$
If it satisfies local consistency then "R-orientable."

* Every manifold is $\mathbb{Z}_2$-orientable & M is orientable $\Rightarrow$ M is R-orientable $\forall R$.

But a non-orientable manifold is R-orientable iff R contains a unit of order 2, which is $2=0$ in R. Thus every mfld is $\mathbb{Z}_2$-orientable.

<u>Thm 3.26</u> Let M be a closed connected n-manifold. Then:

a) If M is R-orientable, then $H_n(M;R) \to H_n(M, M-\{x\}; R) \cong R$ is an isomorphism $\forall x \in M$.

b) If M is not R-orientable, then $H_n(M;R) \to H_n(M, M-\{x\}; R) \cong R$ is injective with image $\{r \in R \mid 2r=0\}$ $\forall x \in M$.

c) $H_i(M;R) = 0$ for $i > n$.

In particular, $H_n(M;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } M \text{ is orientable} \\ 0 & \text{if } M \text{ is non-orientable} \end{cases}$

$H_n(M;\mathbb{Z}_2) = \mathbb{Z}_2$.

<u>Defn:</u> An element of $H_n(M;R)$ whose image in $H_n(M, M-\{x\}; R)$ is a generator for all x is called a "fundamental class" for M. (*) By Thm 3.26 a fundamental class exists if M is closed, R-orientable. Converse is also true, let $\mu \in H_n(M;R)$ be a fund. class & μ_x is the image in $H_n(M, M-\{x\}; R)$. The function $x \mapsto \mu_x$ is then an R-orientation since

$$\begin{tikzcd} H_n(M;R) \arrow[r] \arrow[dr] & H_n(M, M-\{x\}; R) \\ & H_n(M, M-B; R) \arrow[u, "\cong"'] \end{tikzcd} \quad x \in B \underset{\text{open}}{\subseteq} \mathbb{R}$$

M must be compact since μ_x can only be nonzero for x in the image of a cycle representing μ.

For a closed n-mnfld having Δ-cx structure; consider $\mathbb{Z}$ coeff. In simplicial homology a fundamental class must be represented by some $\sum_i k_i \sigma_i$ of the n-simplices σ_i of M. Fundamental class must be mapped to a generator of $H_n(M, M-\{x\})$ $\forall$ so $k_i = \pm 1$. Also $\sum_i k_i \sigma_i$ must be a cycle. So, σ_i & σ_j must share a common $(n-1)$-dim'l face, making $\sum_i k_i \sigma_i$ a cycle is possible iff M is orientable.

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Corollary 3.28: If M is a closed connected n-mnfld, the torsion subgroup of $H_{n-1}(M;\mathbb{Z})$ is trivial if M is orientable & $\mathbb{Z}_2$ if M is nonorientable.

Proof: Closed $\Rightarrow$ homology groups are finitely generated

M orientable, if $H_{n-1}(M;\mathbb{Z})$ has torsion then for some prime p, $H_n(M;\mathbb{Z}_p) = H_n(M)\otimes\mathbb{Z}_p \oplus \operatorname{Tor}(H_{n-1}(M),\mathbb{Z}_p)$

~~Then~~ $H_n(M;\mathbb{Z}_p)$ would be larger than $\mathbb{Z}_p$ coming from $H_n(M)$

M nonorientable; $H_n(M;\mathbb{Z}_m) = \begin{cases} \mathbb{Z}_2 & m=\text{even} \\ 0 & m=\text{odd} \end{cases} \Rightarrow$ Torsion subgroup $\mathbb{Z}_2$.

$H_n(M;G)$ & $H_{n-1}(M;G)$ for a closed, connected n-mnfld M can be explained very nicely by cellular homology when M has a CW-structure with a single n-cell. It produces non-trivial local homology in that dimension. Consider

$$d: C_n(M) \to C_{n-1}(M) \qquad C_n(M)\approx\mathbb{Z}$$

If M is orientable then d must be zero since $H_n(M)\approx\mathbb{Z}$. So $H_{n-1}(M)$ must be free. If M is non-orientable then d takes a generator of $C_n(M)$ to twice a generator α of a $\mathbb{Z}$ summand of $C_{n-1}(M)$ in order for $H_n(M;\mathbb{Z}_p)=\begin{cases}\mathbb{Z}_2 & p=2\\ 0 & \text{o/w}\end{cases}$.

α must be a cycle since 2α is a boundary.

So, the torsion subgroup of $H_n(M;\mathbb{Z})$ must be $\mathbb{Z}_2$ generated by α.

= Cap Product =

X: space, R: ring, define R-bilinear map

$$\frown: C_k(X;R)\times C^\ell(X;R) \longrightarrow C_{k-\ell}(X;R) \text{ for } k\geq \ell \text{ by}$$

$$\sigma\frown\varphi = \varphi(\sigma|_{[v_0,\dots,v_\ell]})\,\sigma|_{[v_\ell,\dots,v_k]}$$

$\sigma:\Delta^k\to X$ & $\varphi\in C^\ell(X;R)$.

Lemma: $\partial(\sigma\frown\varphi) = (-1)^\ell(\partial\sigma\frown\varphi - \sigma\frown\delta\varphi)$

Proof: $\partial(\sigma)\frown\varphi = \sum_{i=0}^{\ell}(-1)^i\varphi(\sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_{\ell+1}]})\sigma|_{[v_{\ell+1},\dots,v_k]}$

$$+ \sum_{i=l+1}^{k} (-1)^i \varphi(\sigma|_{[v_0,\dots,v_l]}) \, \sigma|_{[v_l,\dots,\hat{v}_i,\dots,v_k]}$$

$$\sigma \cap \delta\varphi = \sum_{i=0}^{l+1} (-1)^i \varphi(\sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_{l+1}]}) \, \sigma|_{[v_{l+1},\dots,v_k]}$$

$$\partial(\sigma\cap\varphi) = \sum_{i=l}^{k} (-1)^{i-l} \varphi(\sigma|_{[v_0,\dots,v_l]}) \, \sigma|_{[v_l,\dots,\hat{v}_i,\dots,v_k]}$$

$$\partial\sigma \cap \varphi - \sigma \cap \delta\varphi = -(-1)^{l+1} \varphi(\sigma|_{[v_0,\dots,v_l]}) \, \sigma|_{[v_{l+1},\dots,v_k]} + \sum_{i=l+1}^{k} (-1)^i \varphi(\sigma|_{[v_0,\dots,v_l]}) \, \sigma|_{[v_l,\dots,\hat{v}_i,\dots,v_k]}$$

$$= \sum_{i=l}^{k} (-1)^i \varphi(\sigma|_{[v_0,\dots,v_l]}) \, \sigma|_{[v_l,\dots,\hat{v}_i,\dots,v_k]} = (-1)^l \underbrace{\sum_{i=l}^{k} (-1)^i \dots}_{\partial(\sigma\cap\varphi)}$$

~~$\cap$ is well defined!~~

$\partial(\sigma\cap\varphi) = \pm(\partial\sigma\cap\varphi - \sigma\cap\delta\varphi)$ so cap product of a cycle & a cocycle is a cycle

If $\partial\sigma = 0$ then $\partial(\sigma\cap\varphi) = \pm(\sigma\cap\delta\varphi)$ so cap product of a cycle & coboundary is a boundary.

If $\delta\varphi = 0$ then $\partial(\sigma\cap\varphi) = \pm(\partial\sigma\cap\varphi)$, so cap product of a boundary and a cocycle is a boundary.

So, this induces a map (well-defined)

$$\cap : H_k(X;R) \times H^l(X;R) \xrightarrow{\cap} H_{k-l}(X;R)$$
$$([\sigma],[\varphi]) \longmapsto [\sigma\cap\varphi]$$

Lemma: Cap product satisfies naturality. Given a map $f: X \to Y$ then

$$\begin{array}{ccccc}
H_k(X) & \times & H^l(X) & \xrightarrow{\cap} & H_{k-l}(X) \\
\downarrow f_* & & \uparrow f^* & & \downarrow f_* \\
H_k(Y) & \times & H^l(Y) & \xrightarrow{\cap} & H_{k-l}(Y)
\end{array}$$

$$f_*(\alpha \cap f^*\beta) = f_*(\alpha) \cap \beta$$

Proof:

$$\begin{array}{ccccc}
C_k(X) & \times & C^l(X) & \longrightarrow & C_{k-l}(X) \\
\downarrow f_\# & & \uparrow f^\# & & \downarrow f_\# \\
C_k(Y) & \times & C^l(Y) & \longrightarrow & C_{k-l}(Y)
\end{array}$$

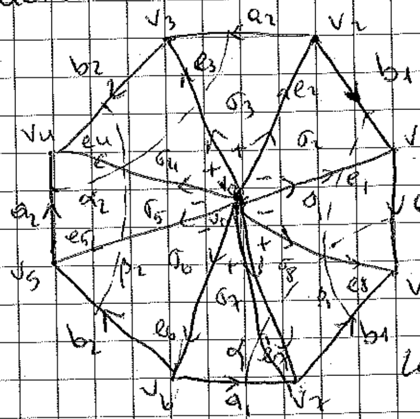
$$\sigma \cap (f^{\#}\varphi) = (f^{\#}\varphi)(\sigma|_{[v_0,\dots,v_\ell]})\,\sigma|_{[v_\ell,\dots,v_k]}$$
$$= \varphi(f\sigma|_{[v_0,\dots,v_\ell]})\,\sigma|_{[v_\ell,\dots,v_k]}$$
$$f_{\#}(\sigma\cap f^{\#}\varphi) = \varphi(f\sigma|_{[v_0,\dots,v_\ell]})\, f\sigma|_{[v_\ell,\dots,v_k]} = (f\sigma)\cap\varphi$$
$$= f_{\#}(\sigma)\cap\varphi \quad \blacksquare$$

<u>Thm 3.30</u> (Poincaré Duality): If M is a closed R-orientable n-mnfld with fundamental class $[M]\in H_n(M;R)$, then the map

$$D: H^k(M;R) \longrightarrow H_{n-k}(M;R) \text{ given by}$$
$$\alpha \longmapsto [M]\cap\alpha$$

is an isomorphism $\forall k$.

<u>Exp: 3.31</u> <u>Surfaces:</u> Let M be a closed orientable surface of genus g, obtained as usual from a $4g$-gon by identifying pairs of edges according to the word $a_1b_1a_1^{-1}b_1^{-1}\dots a_gb_ga_g^{-1}b_g^{-1}$. A Δ-complex structure on M is obtained by by coning off the $4g$-gon to its center.



A fundamental class $[M]$:

$$-\sigma_1-\sigma_2+\sigma_3+\sigma_4-\sigma_5-\sigma_6+\sigma_7+\sigma_8$$

Under the isomorphism $H^1(M)\cong \mathrm{Hom}(H_1(M),\mathbb{Z})$

$$\alpha_i = a_i^* \qquad \beta_i = b_i^*$$

Let $\alpha_i=[\varphi_i]$ $\quad\beta_i=[\psi_i]$ for cocycles φ_i & ψ_i

$$[M]\cap\varphi_1 = (-\sigma_1-\sigma_2+\sigma_3+\sigma_4-\sigma_5-\sigma_6+\sigma_7+\sigma_8)\cap\varphi_1$$
$$= -\underbrace{\sigma_1\cap\varphi_1}_{=0} - \underbrace{\sigma_2\cap\varphi_1}_{=0} + \underbrace{\sigma_3\cap\varphi_1}_{=0} + \underbrace{\sigma_4\cap\varphi_1}_{=0} - \underbrace{\sigma_5\cap\varphi_1}_{=0} - \underbrace{\sigma_6\cap\varphi_1}_{=0}$$
$$+\sigma_7\cap\varphi_1+\sigma_8\cap\varphi_1$$

$$\sigma_1\cap\varphi_1 = \varphi_1(\sigma_1|_{[v_0,v_1]})\cdot\sigma_1|_{[v_1,v_8]} = 0\cdot\sigma_1|_{[v_1,v_8]} = 0$$
$$\sigma_7\cap\varphi_1 = \varphi_1(\sigma_7|_{[v_0,v_6]})\cdot\sigma_7|_{[v_6,v_7]} = 0\cdot\sigma_7|_{[v_6,v_7]} = 0$$
$$\sigma_8\cap\varphi_1 = \varphi_1(\sigma_8|_{[v_0,v_7]})\cdot\sigma_8|_{[v_7,v_8]} = 1\cdot\sigma_8|_{[v_7,v_8]} = b_1$$

Similarly $[M]\cap\varphi_2 = b_2$

$[M]\cap\psi_1 = -a_1 \qquad [M]\cap\psi_2 = -a_2$

Thus
$$H^1(M) \longrightarrow H_1(M)$$
$$\alpha_i \longmapsto [b_i]$$
$$\beta_i \longmapsto -[a_i]$$