Universal Coeff. Thm for Homology:

Thm: There are s.e.s.

$$0 \to H_n(C_*)\otimes G \to H_n(C_*;G) \to \mathrm{Tor}(H_{n-1}(C_*),G) \to 0 \quad \forall n,$$

which are split, not naturally though.

Tor?: Take a free resolution of abelian group ($\mathbb{Z}$-module) H

Which is an exact sequence

$$F_* : \cdots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$$

with F_i's are free abelian.

Given an abelian group G & form a ~~process~~ modified chain complex

$$F_*\otimes G : \cdots \to F_2\otimes G \to F_1 \otimes G \to F_0 \otimes G \to 0$$

$\mathrm{Tor}_n(H,G) := H_n(F_*\otimes G)$

$\mathrm{Tor}_0(H,G) \cong H\otimes G$ (right exact)!

$\mathrm{Tor}_*(H,G) := \mathrm{Tor}_1(H,G)$ Rmk $\ast$ $\mathrm{Tor}_n(H,G)=0$ for abelian groups since $0\to F_1 \hookrightarrow F_0 \twoheadrightarrow H \to 0$

Properties of Tor!

(1) $\mathrm{Tor}(A,B) \cong \mathrm{Tor}(B,A)$

(2) $\mathrm{Tor}(\bigoplus A_i, B) \cong \bigoplus_i \mathrm{Tor}(A_i,B)$

(3) $\mathrm{Tor}(A,B) = 0$ if A or B is free or torsion-free

(4) $\mathrm{Tor}(A,B) \cong \mathrm{Tor}(\mathrm{Torsion}(A), B)$

(5) $\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z}, A) \cong \ker(A \xrightarrow{\times n} A)$

(6) $0 \to B \to C \to D \to 0$ s.e.s. of abelian groups we have

$$0 \to \mathrm{Tor}(A,B) \to \mathrm{Tor}(A,C) \to \mathrm{Tor}(A,D) \to A\otimes B \to A\otimes C \to A\otimes D \to 0.$$

Exp: $G = \mathbb{Q}$ then $\mathrm{Tor}(H_{n-1}(X), \mathbb{Q}) = 0$ so

$$H_n(X;\mathbb{Q}) \cong H_n(X) \otimes \mathbb{Q}$$

so $b_n(X) := \mathrm{rk}\, H_n(X) = \dim_{\mathbb{Q}} H_n(X;\mathbb{Q})$

Exp: $X = T^2$ $G = \mathbb{Z}/4$ $H_1(T^2) = \mathbb{Z} \oplus \mathbb{Z}$

$$H_0(T^2; \mathbb{Z}/4) = H_0(T^2) \otimes \mathbb{Z}/4 = \mathbb{Z} \otimes \mathbb{Z}/4 = \mathbb{Z}/4$$

$$H_1(T^2; \mathbb{Z}/4) = (H_1(T^2) \otimes \mathbb{Z}/4) \oplus \mathrm{Tor}(H_0(T^2), \mathbb{Z}/4) = (\mathbb{Z} \oplus \mathbb{Z}) \otimes \mathbb{Z}/4 = \mathbb{Z}/4 \oplus \mathbb{Z}/4$$

$$H_2(T^2; \mathbb{Z}/4) = (H_2(T^2) \otimes \mathbb{Z}/4) \oplus \mathrm{Tor}(H_1(T^2); \mathbb{Z}/4) = \mathbb{Z}/4$$

Exp: $X = K$ $G = \mathbb{Z}/4$ $H_0(K) = \mathbb{Z}$ $H_1(K) = \mathbb{Z} \oplus \mathbb{Z}/2$ $H_2(K) = 0$

so $$H_2(K; \mathbb{Z}/4) = (H_2(K) \otimes \mathbb{Z}/4) \oplus \mathrm{Tor}(H_1(K), \mathbb{Z}/4)$$

$$= \mathrm{Tor}(\mathbb{Z}, \mathbb{Z}/4) \oplus \mathrm{Tor}(\mathbb{Z}/2, \mathbb{Z}/4) = 0 \oplus \mathbb{Z}/2$$

COHOMOLOGY

Let X be a topological space & G be an abelian group. We will define $H^i(X;G)$ & put a ring structure on $\bigoplus_i H^i(X;G)$ via cup product.

1. **Defn:** (Cohomology of a chain complex)!

Let (C_*, ∂_*) be a chain complex of free abelian group (e.g. $\mathbb{Z}$-modules)

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \qquad (*)$$

We dualize the chain complex $*$; means that apply $\mathrm{Hom}(\_\_, G)$ to it, to get a cochain complex

$$\cdots \xleftarrow{\delta^{n+1}} C^{n+1} \xleftarrow{\delta^n} C^n \xleftarrow{\delta^{n-1}} C^{n-1} \longleftarrow \cdots$$

with
$$C^n := \mathrm{Hom}(C_n, G)$$ where the coboundary map

$\delta^n : C^n \longrightarrow C^{n+1}$ is defined by

$$(\delta^n \psi)(\alpha) := \psi(\partial_{n+1}(\alpha)) \quad \text{for } \psi \in C^n \text{ \& } \alpha \in C_{n+1}.$$

$$\begin{tikzcd}
\beta \arrow[r, mapsto] & \partial_{n+2}(\beta) = \alpha \arrow[r, mapsto] & \partial_{n+1}(\alpha) \\
C_{n+2} \arrow[r, "\partial_{n+2}"] \arrow[drr, dashed] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[dr, dashed, "\delta^n\psi"'] & C_n \arrow[d, "\psi"] \\
& & G
\end{tikzcd}$$

$$\begin{aligned}
(\delta^{n+1} \circ \delta^n)(\psi) &= \delta^{n+1}(\delta^n \psi) \\
&= \delta^n \psi(\partial_{n+2}(\beta)) \\
&= \psi(\partial_{n+1}(\partial_{n+2}(\beta))) \\
&= \psi(\underbrace{\partial_{n+1} \circ \partial_{n+2}}_{=0}(\beta)) \\
&= \psi(0) = 0
\end{aligned}$$

So, this is also a chain cx.

"Chain complex: A chain cx (C_*, ∂_*) is a sequence of abelian groups or modules connected by homomorphisms called boundary operators or differentials $\partial_n : C_n \longrightarrow C_{n-1}$ such that $\partial_n \circ \partial_{n+1} = 0 \quad \forall n.$

Cochain complex: (a variant of chain complex) A cochain complex (C^*, δ^*) is a sequence of abelian groups or modules connected by homomorphisms $\delta^n : C^n \longrightarrow C^{n+1}$ s.t. $\delta^{n+1} \circ \delta^n = 0 \quad \forall n.$"

Defn: The n^{th} cohomology group $H^n(C_*, G)$ with coefficients in G is defined by

$$H^n(C_*; G) := H_n(C^*; \delta^n) := \frac{\ker(\delta : C^n \to C^{n+1})}{\operatorname{Im}(\delta : C^{n-1} \to C^n)}$$

2 - Relation between homology & cohomology!

What is the relation of $H^n(C; G)$ & $\operatorname{Hom}(H_n(C); G)$?

Exp:

$$\begin{array}{ccccccccccc} 0 \to & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{\times 2} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \to 0 & : C_* \\ & \| & & \| & & \| & & \| & & \\ & C_3 & & C_2 & & C_1 & & C_0 & & \end{array}$$

$H_0(C_*) = \mathbb{Z}/0 \cong \mathbb{Z} \qquad H_1(C_*) = \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2$

$H_2(C_*) = 0 \qquad H_3(C_*) = \mathbb{Z}$

Apply $\operatorname{Hom}(C_*; \mathbb{Z})$

$$0 \leftarrow \operatorname{Hom}(\mathbb{Z}, \mathbb{Z}) \xleftarrow{0} \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}) \xleftarrow{\times 2} \operatorname{Hom}(\mathbb{Z}, \mathbb{Z}) \xleftarrow{0} \operatorname{Hom}(\mathbb{Z}, \mathbb{Z}) \leftarrow 0$$

$$\begin{array}{ccccccccccc} 0 \leftarrow & \mathbb{Z} & \xleftarrow{0 = \delta^2} & \mathbb{Z} & \xleftarrow{\times 2 = \delta^1} & \mathbb{Z} & \xleftarrow{0 = \delta} & \mathbb{Z} & \leftarrow 0 & : C^* \\ & \| & & \| & & \| & & \| & & \\ & C^3 & & C^2 & & C^1 & & C^0 & & \end{array}$$

$H^0(C^*) = \mathbb{Z}/0 \cong \mathbb{Z}$ $\quad$ $H^1(C^*) = 0/0 \cong 0$

$H^2(C^*) = \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2$ $\quad$ $H^3(C^*) = \mathbb{Z}/0 \cong \mathbb{Z}$

So $H_2(C_*) = 0$ but $H^2(C^*) = \mathbb{Z}/2$ not equal!

But we can define a homomorphism

$$h : H^n(C_*, G) \longrightarrow \mathrm{Hom}(H_n(C), G) \quad \text{by}$$

$$\varphi \longmapsto \overline{\varphi_0} \quad \text{where } \overline{\varphi_0} \text{ defined as follows:}$$

Remember $Z_n = \mathrm{Ker}(\partial_n)$ cycles & $B_n = \mathrm{Im}(\partial_{n+1})$ boundaries

& we have $\quad 0 \to Z_n \to C_n \to B_{n-1} \to 0$

$$0 \to B_n \to Z_n \to H_n(C) \to 0$$

A class $\varphi \in H^n(C;G)$ is $\varphi : C_n \to G$ s.t. $\delta\varphi = 0$

means $\varphi\partial = 0$ or φ vanishes on B_n i.e. boundaries.

Define $\varphi_0 := \varphi|_{Z_n}$ then this induces a quotient map

$\overline{\varphi_0} : Z_n/B_n \to G$ which is an element of $\mathrm{Hom}(H_n(C); G)$

If $\varphi \in \mathrm{Im}\,\delta$ then $\varphi = \delta\psi = \psi\partial$ then φ is zero on Z_n

so $\varphi_0 = 0$ & $\overline{\varphi_0} = 0$. So this map h is well-defined

& h is a homomorphism.

Moreover h is surjective: Consider $0 \to Z_n \xrightarrow{i} C_n \xrightarrow{\partial} B_{n-1} \to 0$

splits since B_{n-1} is free, hence projective, so $\exists\, p : C_n \to Z_n$

s.t. $p|_{Z_n}$ is identity

$$\begin{tikzcd} C_n \arrow[r, "p"] \arrow[d, "\varphi"'] & Z_n \arrow[dl, "\varphi_0"] \\ G & \end{tikzcd}$$

$\varphi := \varphi_0 \circ p : C_n \to G$

& $\varphi|_{B_n} = 0$

i.e. $\varphi : H_n(C) \to G$, therefore we have a homomorphism

$\mathrm{Hom}(H_n(C), G) \to \mathrm{Ker}\,\delta$.

So h is surjective

Now consider

$$0 \to \ker h \longrightarrow H^n(C;G) \xrightarrow{h} \mathrm{Hom}(H_n(C);G) \to 0$$ is exact.

What is $\ker h$?

Consider

$$\begin{tikzcd}
0 \arrow[r] & Z_{n+1} \arrow[r] \arrow[d, "0"] & C_{n+1} \arrow[r, "\partial"] \arrow[d, "\partial"] & B_n \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & Z_n \arrow[r] & C_n \arrow[r, "\partial"] & B_{n-1} \arrow[r] & 0
\end{tikzcd} \qquad (i)$$

dualize (i) we get

$$\begin{tikzcd}
0 & Z^{n+1} \arrow[l] & C^{n+1} \arrow[l] & B^n \arrow[l, "\delta"'] & 0 \arrow[l] \\
0 & Z^n \arrow[l] \arrow[u, "0"] & C^n \arrow[l] \arrow[u, "\delta"] & B^{n-1} \arrow[l, "\delta"'] \arrow[u, "0"] & 0 \arrow[l]
\end{tikzcd} \qquad (ii)$$

$i_n^*(\alpha)$

Since rows of (i) are split & dual of a split s.e.s. is split $\left(\mathrm{Hom}(A \oplus B, G) \cong \mathrm{Hom}(A,G) \oplus \mathrm{Hom}(B,G)\right)$ the rows of (ii) are also exact

We can consider (ii) a a part of s.e.s. of chain complexes. Since coboundary maps in Z^n & B^n are zero the associated long exact sequence is

$$\cdots \leftarrow B^n \xleftarrow{i_n^*} Z^n \leftarrow H^n(C;G) \leftarrow B^{n-1} \xleftarrow{i_{n-1}^*} Z^{n-1} \leftarrow \cdots \qquad (iii)$$

where $i_n : B_n \to Z_n$.

We can get ~~break~~ (iii) into s.e.s.

$$0 \leftarrow \mathrm{Ker}\, i_n^* \leftarrow H^n(C;G) \leftarrow \mathrm{Coker}\, i_{n-1}^* \leftarrow 0 \quad (iv)$$

Since the elements of $\mathrm{Ker}\, i_n^*$ are homomorphisms $Z_n \to G$ that vanish on B_n, & same as homomorphisms $Z_n/B_n \to G$

$\therefore$ $\mathrm{Ker}\, i_n^* = \mathrm{Hom}(H_n(C), G)$ So $H^n(C;G) \to \mathrm{Ker}\, i_n^*$ is the map h we consider earlier.

Thus we can rewrite (iv) as a split s.e.s.

$$0 \to \operatorname{Coker} i_{n-1}^* \longrightarrow H^n(C;G) \xrightarrow{h} \operatorname{Hom}(H_n(C), G) \to 0 \qquad (v)$$

Now, what is $\operatorname{Coker} i_{n-1}^*$? We have s.e.s.

$$0 \to B_{n-1} \xrightarrow{i_{n-1}} Z_{n-1} \longrightarrow H_{n-1}(C) \to 0$$

dualize

$$0 \leftarrow B^{n-1} \xleftarrow{i_{n-1}^*} Z^{n-1} \longleftarrow \operatorname{Hom}(H_{n-1}(C), G) \leftarrow 0$$

if this were exact, then i_{n-1}^* would be surjective & $\operatorname{coker} i_{n-1}^* = 0$.

But in general this is not always the case:

e.g. $0 \to \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \to \mathbb{Z}/n \to 0$ apply $\operatorname{Hom}(-, \mathbb{Z})$

we get $0 \leftarrow \mathbb{Z} \xleftarrow{\times n} \mathbb{Z} \leftarrow 0 \leftarrow 0$

not exact!

In general, if an abelian group H has a free resolution then we have $(*)$ $\cdots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$ with each F_i free and $(*)$ is exact. If we dualize i.e. apply $\operatorname{Hom}(-, G)$ we may lose exactness, but we get a (co)chain complex

$$\cdots \leftarrow F_2^* \xleftarrow{f_2^*} F_1^* \xleftarrow{f_1^*} F_0^* \xleftarrow{f_0^*} H^* \leftarrow 0$$

which is not necessarily exact but a cx. $\operatorname{Ext}^n(H, G) := \operatorname{Ker} f_{n+1}^* / \operatorname{Im} f_n^*$

So since B_{n-1} & Z_{n-1} are free then

$$0 \to B_{n-1} \xrightarrow{i_{n-1}} Z_{n-1} \to H_{n-1}(C) \to 0 \quad \text{is a}$$

free resolution of $H_{n-1}(C)$ so

$$0 \leftarrow \operatorname{Coker} i_{n-1}^* \leftarrow B^{n-1} \xleftarrow{i_{n-1}^*} Z^{n-1} \longleftarrow \operatorname{Hom}(H_{n-1}(C), G) \leftarrow 0$$

is now exact! So $\operatorname{Coker} i_{n-1}^* = \operatorname{Ext}^1(H_{n-1}(C), G) = \operatorname{Ext}(\ldots)$

Therefore: we get

Thm: (Universal Coefficient Theorem for Cohomology)

If a chain cx C of free abelian groups has homology groups $H_n(C)$, then the cohomology groups $H^n(C;G)$ of the cochain complex $Hom(C_n, G)$ are determined by split exact sequences

$$0 \to Ext(H_{n-1}(C), G) \to H^n(C;G) \xrightarrow{h} Hom(H_n(C), G) \to 0 .$$ ∎

Properties of Ext:

(1) $Ext(H \oplus H', G) \cong Ext(H, G) \oplus Ext(H', G)$

(2) $Ext(H, G) = 0$ if H is free

(3) $Ext(\mathbb{Z}/n, G) \cong G/nG$

Corollary 1 - If the homology groups H_n & H_{n-1} of a chain cx C of free abelian groups are finitely generated with torsion subgroups $T_n \subset H_n$ & $T_{n-1} \subset H_{n-1}$ then

$$H^n(C;\mathbb{Z}) \cong (H_n/T_n) \oplus T_{n-1}$$

Corollary 2 - If a chain map between chain complexes of free abelian groups induces an isomorphism on homology then it induces an isomorphism on cohomology groups with coeff. in G

(Five-Lemma + Naturality).

3- Cohomology of Spaces:

Suppose X is a topological space with singular chain complex $(C_\bullet(X), \partial_\bullet)$. The group of singular n-cochains of X are defined as

$$C^n(X;G) := Hom(C_n(X); G)$$ "functions from singular n-chains to G"

$\varphi \in C^n(X;G) \Rightarrow \varphi(\sigma) \in G$ where $\sigma : \Delta^n \to X$ n-chain.

The coboundary map $\delta^n : C^n(X;G) \to C^{n+1}(X;G)$ where $\delta = \partial^*$

for $\varphi \in C^n(X;G)$ $\quad \delta\varphi : C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\varphi} G$ $\quad$ i.e $\delta\varphi = \varphi\partial_{n+1}$

$$\delta\varphi(\sigma) = \sum_i^{n+1}(-1)^i \varphi(\sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_{n+1}]}) \quad \text{for } \sigma : \Delta^{n+1} \to X$$

$\delta^2 = 0$ so $(C^n(X;G), \delta)$ is a (co)chain cx.

$$H^n(X;G) = \ker \delta^{n+1} / \operatorname{Im} \delta^n$$

$\operatorname{Ker} \delta^{n+1}$ "n-cocycles" $\quad \operatorname{Im} \delta^{n-1}$: "n-coboundaries"

If $\varphi \in \operatorname{Ker} \delta$ then $\delta\varphi = 0 \Rightarrow \varphi\partial = 0$ or φ is 0 on boundaries.

Since $C_n(X)$ are free by U.C.T. we have split s.e.s.

$$0 \to \operatorname{Ext}(H_{n-1}(X), G) \longrightarrow H^n(X;G) \to \operatorname{Hom}(H_n(X), G) \to 0.$$

$$\operatorname{Hom}(H_n(X), G) \oplus \operatorname{Ext}(H_{n-1}(X), G).$$

RMK:

1. $H^n(X;G) \cong \operatorname{Hom}(H_n(X), G) \oplus \operatorname{Ext}(H_{n-1}(X), G)$
2. $H^0(X;G) \cong \operatorname{Hom}(H_0(X), G)$
3. $H^1(X;G) \cong \operatorname{Hom}(H_1(X), G) \oplus \underbrace{\operatorname{Ext}(\overset{\text{free}}{H_0(X)}, G)}_{0} \cong \operatorname{Hom}(H_1(X), G)$
4. If G is a ~~free~~ module over PID then U.C.T holds
5. If G is a field i.e. $G = \mathbb{Z}/p$ or $\mathbb{Q}$ then

$$H^n(X;F) \cong \operatorname{Hom}_F(H_n(X), F) \cong H_n(X)^*$$

Exp: ① Let X be a point

$$H^i(X;G) = \operatorname{Hom}(H_i(X), G) \oplus \operatorname{Ext}(H_{i-1}(X), G)$$

$$H_i(X) = \begin{cases} \mathbb{Z}, & i = 0 \\ 0, & o/w \end{cases}$$

$$H^i(X;G) \cong \operatorname{Hom}(H_i(X), G) = \begin{cases} G & ; i = 0 \\ 0 & , o/w \end{cases}$$

② $X = S^n$ $\quad H_i(X) = \begin{cases} \mathbb{Z} & ; i = 0, n \\ 0 & ; o/w \end{cases} \Rightarrow H^i(S^n;G) = \begin{cases} G & ; i = 0 \text{ or } n \\ 0 & ; o/w. \end{cases}$

Reduced cohomology groups:

$$\cdots \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

$$\varepsilon\left(\sum_i n_i x_i\right) = \sum_i n_i \qquad \text{after dualizing}$$

$$\cdots \xleftarrow{\delta^1} C^1(X;G) \xleftarrow{\delta^0} C^0(X;G) \xleftarrow{\varepsilon^*} G \leftarrow 0$$

since $\varepsilon\partial_1 = 0$ $\quad \delta^0\varepsilon^* = 0$ the homology of this augmented cochain cx is the reduced cohomology of X with coeff. in G, & denoted by $\tilde{H}^i(X;G)$.

$$\tilde{H}^i(X;G) = H^i(X;G) \quad \text{if } i>0$$

$$\tilde{H}^0(X;G) = \mathrm{Hom}(\tilde{H}_0(X), G) \quad \text{by U.C.T}$$

Relative cohomology groups:

(X,A) be a pair

$$C^n(X,A;G) := \mathrm{Hom}(C_n(X,A);G)$$

" function from n-simplices in X to G that vanish on simplices in A "

$$C^n(X,A;G) \hookrightarrow C^n(X;G)$$

$$\delta : C^n(X,A;G) \longrightarrow C^{n+1}(X,A;G) \qquad \delta^2 = 0$$

So $H^n(X,A;G)$ can be defined.

If we dualize $\quad 0 \to C_n(A) \xrightarrow{i} C_n(X) \xrightarrow{J} C_n(X,A) \to 0$

we get $\quad 0 \leftarrow C^n(A;G) \xleftarrow{i^*} C^n(X;G) \xleftarrow{J^*} C^n(X,A;G) \leftarrow 0$

is exact! By taking associated l.e.s. we get

$$\cdots \longrightarrow H^n(X,A;G) \xrightarrow{J^*} H^n(X;G) \xrightarrow{i^*} H^n(A;G) \xrightarrow{\delta = \partial^* \text{ connecting homom.}} H^{n+1}(X,A;G) \to \cdots$$

Induced homomorphisms:

If $f: X \to Y$ is a cont. map then we have induced chain maps

$$f_\# : C_n(X) \to C_n(Y)$$
$$(\sigma: \Delta_n \to X) \longmapsto (f\circ\sigma : \Delta_n \xrightarrow{\sigma} X \xrightarrow{f} Y) \quad \text{satisfying } f_\#\partial = \partial f_\#$$

Dualize $f_\#$ we get $\quad f^\# : C^n(Y;G) \longrightarrow C^n(X;G)$ with

$$f^\#(\varphi) = \varphi(f_\#) \ \& \ \delta f^\# = f^\#\delta$$

So we get induced homomorphism on cohomology groups

$f^*: H^n(Y,G) \to H^n(X,G)$ "contravariant functor"

<u>Homotopy Invariance:</u> Thm: If $f \simeq g: (X,A) \to (Y,B)$ then

$f^* = g^*: H^n(Y,B;G) \to H^n(X,A;G)$.

for a chain homotopy $P: C_n(X,A) \to C_{n+1}(Y,B)$

satisfying $f_\# - g_\# = P\partial + \partial P$

by dualizing $f^\# - g^\# = \delta P^* + P^*\delta$ so $f^* = g^*$.

<u>Corollary:</u> If $f: X \to Y$ is a h.e. then f^* is an isomorphism.

<u>e.g.</u> $H^i(\mathbb{R}^n; G) = \begin{cases} G & ; i=0 \\ 0 & ; o/w \end{cases}$ since $\mathbb{R}^n \simeq *$

<u>Excision:</u> $Z \subset A \subset X$ with $\overline{Z} \subseteq \mathrm{int}(A)$.

$i: (X \setminus Z, A \setminus Z) \hookrightarrow (X,A)$

$i^*: H^n(X,A;G) \to H^n(X\setminus Z, A\setminus Z; G)$ is an isom $\forall n$.

naturality + five Lemma + U.C.T.

<u>Mayer-Vietoris sequence:</u>

<u>Thm:</u> Let X be a top. space, A & B be subsets of X s.t.

$X = \mathrm{int}(A) \cup \mathrm{int}(B)$ then $\exists$ a l.e.s.

$$\cdots \to H^n(X;G) \xrightarrow{\psi} H^n(A;G) \oplus H^n(B;G) \xrightarrow{\phi} H^n(A\cap B;G) \to H^{n+1}(X;G) \to \cdots$$

(Comes from s.e.s. $0 \to C^n(A+B;G) \xrightarrow{\psi} C^n(A;G) \oplus C^n(B;G) \xrightarrow{\phi} C^n(A\cap B;G) \to 0$)

Rmk: Cellular & Simplicial cohomology are defined similarly! They're all isomorphic.

<u>Exp:</u> ① $X = \mathbb{R}P^2$ it has one cell in each dimension so cellular chain cx is

$$0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

apply $\mathrm{Hom}(-,\mathbb{Z})$

$$0 \leftarrow \mathbb{Z} \xleftarrow{\times 2} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \leftarrow 0$$

$$H^i(\mathbb{RP}^2;\mathbb{Z}) = \begin{cases} \mathbb{Z} & i=0 \\ 0 & i=1 \\ \mathbb{Z}/2\mathbb{Z} & i=2 \\ 0 & o/w \end{cases}$$

what about $H^i(\mathbb{RP}^2;\mathbb{Z}/2)$?
apply $\mathrm{Hom}(-,\mathbb{Z}/2)$

$$0 \leftarrow \mathbb{Z}/2 \xleftarrow{0} \mathbb{Z}/2 \xleftarrow{0} \mathbb{Z}/2 \leftarrow 0$$

$$H^i(\mathbb{RP}^2,\mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 ; & i=0,1,2 \\ 0 & o/w. \end{cases}$$

(2) K: Klein Bottle $H^*(K;\mathbb{Z}/3)$?

$$0 \to \mathbb{Z} \xrightarrow{(2,0)} \mathbb{Z}\oplus\mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

apply $\mathrm{Hom}(-,\mathbb{Z}/3)$ $\quad 0 \leftarrow \mathbb{Z}/3 \xleftarrow{(2,0)} \mathbb{Z}/3 \oplus \mathbb{Z}/3 \xleftarrow{0} \mathbb{Z}/3 \leftarrow 0$

$$H^i(K;\mathbb{Z}/3) = \begin{cases} \mathbb{Z}/3 & i=0,1 \\ 0 & o/w \end{cases}$$

Do it by U.C.T.

= CUP PRODUCT IN COHOMOLOGY =

Motivation: Let $X=\mathbb{CP}^2$ & $Y=S^2\vee S^4$ as CW-cx they have 1 zero cell, 1 two-cell & 1 four-cell so they both have same cellular chain cx.

$$0 \to \overset{4}{\mathbb{Z}} \xrightarrow{0} \overset{3}{0} \xrightarrow{0} \overset{2}{\mathbb{Z}} \xrightarrow{0} \overset{1}{0} \xrightarrow{0} \overset{0}{\mathbb{Z}} \to 0$$

$\therefore$ X & Y have same homology & cohomology groups.
Note that they also have the same fund. groups. $\pi_1(X)=\pi_1(Y)=1$.
Are X & Y homotopy equiv.?

Defn: Let X be a topological space & fix a coeff. ring R $(\mathbb{Z},\mathbb{Z}/n,\mathbb{Q})$
Let $\phi\in C^k(X;R)$ & $\psi\in C^l(X;R)$ The cup product $\phi\cup\psi\in C^{k+l}(X;R)$ is defined by

$$(\phi\cup\psi)(\sigma:\Delta_{k+l}\to X) = \phi(\sigma|_{[v_0,\dots,v_k]})\cdot\psi(\sigma|_{[v_k,\dots,v_{k+l}]})$$

multip. in R.

<u>Lemma</u>: $\delta(\phi \cup \psi) = \delta\phi \cup \psi + (-1)^k \phi \cup \delta\psi$

<u>Proof</u>: Let $\sigma : \Delta^{k+l+1} \to X$

$$(\delta\phi \cup \psi)(\sigma) = \sum_{i=0}^{k+1} (-1)^i \phi(\sigma|_{[v_0,\dots,\hat{v_i},\dots,v_{k+1}]}) \cdot \psi(\sigma|_{[v_{k+1},\dots,v_{k+l+1}]})$$

$$+ \; (-1)^k (\phi \cup \delta\psi)(\sigma) = \sum_{i=k}^{k+l+1} (-1)^i \phi(\sigma|_{[v_0,\dots,v_k]}) \cdot \psi(\sigma|_{[v_k,\dots,\hat{v_i},\dots,v_{k+l+1}]})$$

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the last term of the first sum cancels the first term of the second

$$\delta(\phi \cup \psi)(\sigma) = (\phi \cup \psi)(\partial\sigma) \qquad \partial\sigma = \sum_{i=0}^{k+l+1} (-1)^i \sigma|_{[v_0,\dots,\hat{v_i},\dots,v_{k+l+1}]}$$

<u>Corollary 1</u>: The cup product of two cocycles is again a cocycle

i.e. if $\delta\phi = 0$ & $\delta\psi = 0$ then

$$\delta(\phi \cup \psi) = \underset{0}{\underbrace{\delta\phi}} \cup \psi + (-1)^k \phi \cup \underset{0}{\underbrace{\delta\psi}} = 0$$

<u>Corollary 2</u>: If one of ϕ or ψ is a cocycle & the other a coboundary then $\phi \cup \psi$ is a coboundary.

<u>Proof</u>: Say $\delta\phi = 0$ & $\psi = \delta\eta$

Then $\phi \cup \psi = \phi \cup \delta\eta = \pm\delta(\phi \cup \eta)$

Similarly if $\delta\psi = 0$ & $\phi = \delta\eta$ then $\phi \cup \psi = \delta\eta \cup \psi = \delta(\eta \cup \psi)$.

Therefore we get an induced map on cohomology

$$H^k(X;R) \times H^l(X;R) \xrightarrow{\cup} H^{k+l}(X;R)$$

dist. assoc., if R has identity then $\exists$ is an identity $1 \in H^0(X;R)$

1: 0-cocycle taking the value 1 on each singular 0-simplex.

Consider $\bigoplus_i H^i(X;R)$ graded ring with cup product.

<u>Exp:</u> $X = \mathbb{RP}^2$

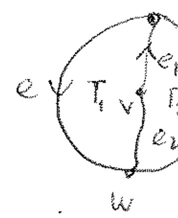
$$H^i(\mathbb{RP}^2; \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & \text{for } i = 0, 1, 2 \\ 0 & \text{o/w} \end{cases}$$

Let $\alpha \in H^1(\mathbb{RP}^2; \mathbb{Z}/2) = \mathbb{Z}/2$ be the generator. Consider

$$\alpha^2 := \alpha \cup \alpha \in H^2(\mathbb{RP}^2; \mathbb{Z}/2)$$

<u>Claim:</u> $\alpha^2 \neq 0$

Consider



Since α is generator of $H^1(\mathbb{RP}^2, \mathbb{Z}/2) \cong \mathrm{Hom}(H_1(\mathbb{RP}^2), \mathbb{Z}/2)$

so $\alpha = [\phi]$ where $\phi : C_1(\mathbb{RP}^2) \to \mathbb{Z}/2$ is a cocycle s.t. $\phi(e) = 1$

where e is the generator of $H_1(\mathbb{RP}^2)$.

$$0 = \delta\phi(T_1) = \phi(\partial T_1) = \phi(e_1 + e - e_2) = \phi(e_1) + \phi(e) - \phi(e_2)$$

$$0 = \delta\phi(T_2) = \phi(\partial T_2) = \phi(e_2) + \phi(e) - \phi(e_1)$$

since $\phi(e) = 1$ w.l.o.g. we can take $\phi(e_1) = 1$ & $\phi(e_2) = 0$

$\alpha^2 = \alpha \cup \alpha$ is represented by $[\phi \cup \phi]$; since $T_1 : [vww] \to \mathbb{RP}^2$

$$\phi \cup \phi(T_1) = \phi(T_1|_{[v,w]}) \cdot \phi(T_1|_{[w,w]})$$

$$= \phi(e_1) \cdot \phi(e) = 1$$

& $\phi \cup \phi(T_2) = \phi(e_2) \cdot \phi(e) = 0$

The generator of $H_2(\mathbb{RP}^2; \mathbb{Z}/2)$ is $T_1 + T_2$ & we have

$$(\phi \cup \phi)(T_1 + T_2) = (\phi \cup \phi)(T_1) + (\phi \cup \phi)(T_2) = 1 + 0 = 1$$

so $\alpha^2 = [\phi \cup \phi]$ which is the generator of $H^2(\mathbb{RP}^2; \mathbb{Z}/2)$. ∎

Lemma: $\alpha \cup \beta = (-1)^{kl} \beta \cup \alpha$

Lemma: The cup product is functorial, i.e. for a map

$f: X \to Y$ the induced maps satisfy

$$f^*: H^i(Y;R) \to H^i(X;R) \qquad f^*(\alpha \cup \beta) = f^*(\alpha) \cup f^*(\beta)$$

Proof: For $\phi \in C^k(X;R)$ & $\psi \in C^l(X;R)$ we have

$$\begin{aligned} f^\#(\phi) \cup f^\#(\psi)(\sigma: \Delta_{k+l} \to X) &= (f^\# \phi)(\sigma|_{[v_0,\dots,v_k]}) \cdot f^\#(\psi)(\sigma|_{[v_k,\dots,v_{k+l}]}) \\ &= \phi((f_\# \sigma)|_{[v_0,\dots,v_k]}) \cdot \psi((f_\# \sigma)|_{[v_k,\dots,v_{k+l}]}) \\ &= (\phi \cup \psi)(f_\# \sigma) \\ &= (f^\#(\phi \cup \psi))(\sigma). \end{aligned}$$

∎

Defn: A graded ring is a ring A with sum decomposition $A = \bigoplus_k A_k$ where A_k are additive subgroups so that the multiplication of A takes $A_k \times A_l$ to A_{k+l}.

Elements of A_k are called of degree k,

Defn: The cohomology ring of a top. space X is the graded ring

$$H^*(X;R) := \left(\bigoplus_{k \geq 0} H^k(X;R), \cup\right)$$

Corollary: If $f: X \to Y$ is a cont. map then we get an induced ring homomorphism

$$f^*: H^*(Y;R) \to H^*(X;R).$$

Expls:

$$H^*(\coprod_\alpha X_\alpha; R) \xrightarrow{\cong} \prod_\alpha H^*(X_\alpha; R)$$

$$\tilde{H}^*(\bigvee_\alpha X_\alpha; R) \longrightarrow \prod_\alpha \tilde{H}^*(X_\alpha; R)$$

Exp: $H^*(\mathbb{R}P^2; \mathbb{Z}/2) = \{a_0 + a_1\alpha + a_2\alpha^2 \mid a_i \in \mathbb{Z}/\mathbb{Z}\}$

$= \mathbb{Z}/2[\alpha]/(\alpha^3) \quad \alpha \in H^1(\mathbb{R}P^2, \mathbb{Z}/2)$

Exp: $H^*(S^n; \mathbb{Z}) = \mathbb{Z}[\alpha]/(\alpha^2)$

If α is a generator of $H^n(S^n; \mathbb{Z})$ then $\alpha \cup 1$ or $\alpha \cup \alpha$

but $\alpha \cup \alpha \in H^{2n}(S^n; \mathbb{Z}) = 0$. Hence $\alpha^2 = 0$.

Exp: Let $X = \mathbb{C}P^2$ & $Y = S^2 \vee S^4$ have same homology & cohomology groups.

Compute $H^*(X; \mathbb{Z})$ & $H^*(Y; \mathbb{Z})$

$H^*(\mathbb{C}P^2; \mathbb{Z}) = \mathbb{Z}[\beta]/(\beta^3)$ where β is the generator of $H^2(\mathbb{C}P^2; \mathbb{Z})$

& $\tilde{H}^*(S^2 \vee S^4; \mathbb{Z}) \cong \tilde{H}^*(S^2; \mathbb{Z}) \oplus \tilde{H}^*(S^4; \mathbb{Z})$

$H^*(S^2; \mathbb{Z}) = \mathbb{Z}[\alpha]/(\alpha^2)$ & $H^*(S^4; \mathbb{Z}) = \mathbb{Z}[\gamma]/\gamma^2$

& $\alpha^2 = 0 = \gamma^2 \quad \alpha \cup \gamma \in H^6(S^2 \vee S^4; \mathbb{Z}) = 0$

both X & Y have a degree 2 cohomology generators

$\beta \cup \beta \in H^4(\mathbb{C}P^2; \mathbb{Z})$ & $\beta^2 \neq 0$. However $\alpha^2 \in H^4(S^2; \mathbb{Z}) = 0$

so ~~cohomology~~ rings of X & Y are not isomorphic, hence X & Y are not homotopy equivalent.

Exp:

1. $H^*(\mathbb{R}P^n; \mathbb{Z}/2) \cong \mathbb{Z}/2[\alpha]/(\alpha^{n+1})$; $|\alpha| = 1$.
2. $H^*(\mathbb{R}P^\infty; \mathbb{Z}/2) \cong \mathbb{Z}/2[\alpha]$; $|\alpha| = 1$.
3. $H^*(\mathbb{C}P^n; \mathbb{Z}) \cong \mathbb{Z}[\beta]/(\beta^{n+1})$; $|\beta| = 2$
4. $H^*(\mathbb{C}P^\infty; \mathbb{Z}) \cong \mathbb{Z}[\beta]$; $|\beta| = 2$

Künneth Formula:

1 – Cross Product:

$X = S^2 \times S^3$ & $Y = S^2 \vee S^3 \vee S^5$

Both has CW complexes with cells $\{e^0, e^2, e^3, e^5\}$

so chain complex

$$0 \to \mathbb{Z} \to 0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0 \to \mathbb{Z} \to 0$$

Are they X & Y homotopy equivalent?

$$\tilde{H}^*(S^2 \vee S^3 \vee S^5; \mathbb{Z}) \cong \tilde{H}^*(S^2; \mathbb{Z}) \oplus \tilde{H}^*(S^3; \mathbb{Z}) \oplus \tilde{H}^*(S^5; \mathbb{Z})$$

$$\mathbb{Z}[\alpha]/(\alpha^2) \oplus \mathbb{Z}[\beta]/(\beta^2) \oplus \mathbb{Z}[\gamma]/(\gamma^2)$$

with $|\alpha| = 2$ $|\beta| = 3$ $|\gamma| = 5$

& $\alpha \cup \beta = 0$: $p: S^2 \vee S^3 \vee S^5 \to S^2 \vee S^3$ rest. map.

$$p^*: H^{2(3)}(S^2 \vee S^3) \longrightarrow H^{2(3)}(S^2 \vee S^3 \vee S^5)$$

then $\alpha = p^*(\tilde{\alpha})$ & $\beta = p^*(\tilde{\beta})$ so

$$\alpha \cup \beta = p^*(\tilde{\alpha}) \cup p^*(\tilde{\beta}) = p^*(\underbrace{\tilde{\alpha} \cup \tilde{\beta}}_{= 0}) = 0$$

<u>Thm</u>: Let R be a comm. ring & $\alpha \in H^k(X, A; R)$ & $\beta \in H^l(X, A; R)$

then $\alpha \cup \beta = (-1)^{kl} \beta \cup \alpha$

So, $H^*(X, A; R)$ is a graded commutative ring.

<u>Corollary</u>: If $\alpha \in H^*(X; R)$ is of odd degree & if $H^*(X, R)$ has no elements of order two then $\alpha \cup \alpha = 0$

$\alpha \cup \alpha = (-1)^{n \cdot n} \alpha \cup \alpha$ n is odd so $\alpha \cup \alpha = -\alpha \cup \alpha \Leftrightarrow$

$2\alpha \cup \alpha = 0$

but no involution $\Leftrightarrow \alpha \cup \alpha = 0$.

<u>Defn:</u> (Cross Product)

Let X, Y be top spaces $p: X\times Y \to X$ & $q: X\times Y \to Y$ be projections

By using cohomology we define

$$H^*(X;R)\times H^*(Y;R) \xrightarrow{\times} H^*(X\times Y;R) \quad \text{by}$$
$$(a,b) \longmapsto a\times b := p^*(a)\cup q^*(b)$$

cross product is linear in each variable so bilinear map so they're rarely homomorphisms. But by universal property of tensor products we can replace it by

$$H^*(X;R)\otimes_R H^*(Y;R) \xrightarrow{\times} H^*(X\times Y;R)$$

so by defn. we obtain

$$\times(a\otimes b) := a\times b \,.$$

If we put a ring str. on $H^*(X;R)\otimes_R H^*(Y;R)$ by

$$(a\otimes b)\cdot(c\otimes d) = (-1)^{\deg(b)\cdot\deg(c)}(a\overset{\cup}{c}\otimes b\overset{\cup}{d})$$

we have

$$\times((a\otimes b)\cdot(c\otimes d)) = (-1)^{\deg(b)\cdot\deg(c)}\times(ac\otimes bd)$$
$$= (-1)^{\deg(b)\deg(c)}\, ac\times bd$$
$$= (-1)^{\deg(b)\deg(c)}\, p^*(a\cup c)\cup q(b\cup d)$$
$$= (-1)^{\deg(b)\deg(c)}\, p^*(a)\cup p^*(c)\cup q^*(b)\cup q^*(d)$$
$$= (-1)^{|b||c|}(-1)^{|b||c|}\, \underbrace{p^*(a)\cup q^*(b)}\cup\underbrace{p^*(c)\cup q^*(d)}$$
$$= a\times b\cup c\times d$$
$$= \times(a\otimes b)\cup\times(c\otimes d) \qquad \text{homomorphism!}$$

<u>Thm:</u> $H^*(X;R)\otimes_R H^*(Y;R) \longrightarrow H^*(X\times Y;R)$ is an ring isomorphism of rings if X & Y are CW-cxs & $H^k(Y;R)$ is fin. gen. free R-module $\forall k$.

<u>~~Exer~~ Moreover:</u> $$H^n(X\times Y;R)\cong \bigoplus_{i+j=n} H^i(X;R)\otimes_R H^j(Y;R)$$

<u>Exp:</u> 1 - $$H^*(\mathbb{R}P^\infty\times \mathbb{R}P^\infty;\mathbb{Z}/2)\cong H^*(\mathbb{R}P^\infty;\mathbb{Z}/2)\otimes H^*(\mathbb{R}P^\infty;\mathbb{Z}/2)$$
$$\mathbb{Z}/2[\alpha]\otimes_{\mathbb{Z}/2}\mathbb{Z}/2[\beta]\cong \mathbb{Z}/2[\alpha,\beta]$$

2 - $$H^*(S^2\times S^3;\mathbb{Z})\cong H^*(S^2;\mathbb{Z})\otimes_{\mathbb{Z}} H^*(S^3;\mathbb{Z})$$

Let $a\in H^*(S^2;\mathbb{Z})$; $|a|=2$ & $b\in H^*(S^3;\mathbb{Z})$; $|b|=3$

then $\times(a\otimes 1)=a\times 1$ & $\times(1\otimes b)=1\times b$ 1 identity of corr. coh. ring

so $H^*(S^2\times S^3;\mathbb{Z})$ has generators of deg 2 & deg 3 ; res.

Moreover $(a\times 1)\cup(1\times b)=\times(a\otimes 1)\cup\times(1\otimes b)=\times(a\otimes b)=a\times b$

a generator of degree 5 in $H^*(S^2\times S^3;\mathbb{Z})$

<u>Rmk:</u> We can represent these by exterior algebras over R

$\Lambda_R[\alpha_1,\alpha_2,\dots]$ is generated by

$\alpha_{i_1}\cdots\alpha_{i_k}$ $\quad i_1<i_2<\dots<i_k$

multiplication def. by $\alpha_i\alpha_j=-\alpha_j\alpha_i$ if $i\neq j$

& $\alpha_i^2=0$

empty product is 1.

<u>e.g.</u> $$H^*(S^3\times S^5\times S^7;\mathbb{Z})\cong \Lambda_{\mathbb{Z}}[a_3,a_5,a_7]$$

$$H^*(S^3\times S^5\times S^7;\mathbb{Z})\cong H^*(S^3;\mathbb{Z})\otimes_{\mathbb{Z}} H^*(S^5;\mathbb{Z})\otimes_{\mathbb{Z}} H^*(S^7;\mathbb{Z})$$

Let $\alpha_i\in H^*(S^i;\mathbb{Z})$ $\quad i=3,5,7$

$a_3=\times(\alpha_3\otimes 1\otimes 1)$

$a_5=\times(1\otimes\alpha_5\otimes 1)$

$a_7=\times(1\otimes 1\otimes\alpha_7)$

$$a_3^2 = x(\alpha_3 \otimes 1 \otimes 1) \cup x(\alpha_3 \otimes 1 \otimes 1)$$
$$= x[(\alpha_3 \otimes 1 \otimes 1) \cdot (\alpha_3 \otimes 1 \otimes 1)]$$
$$= x(\alpha_3^2 \otimes 1 \otimes 1) = 0$$

Similar results for a_5^2 & a_7^2 holds.

$$a_3 a_5 = x(\alpha_3 \otimes 1 \otimes 1) \cup x(1 \otimes \alpha_5 \otimes 1)$$
$$= x[(\alpha_3 \otimes 1 \otimes 1) \cdot (1 \otimes \alpha_5 \otimes 1)]$$
$$= (-1)^{0.0} x(\alpha_3 \otimes \alpha_5 \otimes 1) = x(\alpha_3 \otimes \alpha_5 \otimes 1)$$

$$a_5 a_3 = x(1 \otimes \alpha_5 \otimes 1) \cup x(\alpha_3 \otimes 1 \otimes 1)$$
$$= x[(1 \otimes \alpha_5 \otimes 1) \cdot (\alpha_3 \otimes 1 \otimes 1)]$$
$$= (-1)^{3.5} x(\alpha_3 \otimes \alpha_5 \otimes 1) = -a_3 a_5$$

So $H^*(S^3 \times S^5 \times S^7; \mathbb{Z}) \cong \Lambda_{\mathbb{Z}}[a_3, a_5, a_7]$

Rmk: This holds for product of any odd dim'l spheres.

Exp: $H^*(\mathbb{R}P^\infty \times \mathbb{R}^\infty; \mathbb{Z}/2) = H^*(\mathbb{R}P^\infty; \mathbb{Z}/2) \otimes_{\mathbb{Z}/2} H^*(\mathbb{R}P^\infty; \mathbb{Z}/2)$

$$= \mathbb{Z}/2[\alpha] \otimes_{\mathbb{Z}/2} \mathbb{Z}/2[\beta]$$
$$\cong \mathbb{Z}/2[\alpha, \beta] \quad \text{where } |\alpha| = |\beta| = 1$$
$$\alpha \cup \beta = \beta \cup \alpha$$

In general; let R be a PID

Given two chain complexes (C_*, ∂_*) & (C'_*, ∂'_*)

define $(C \otimes C')_*$ by $(C \otimes C')_n = \bigoplus_{p=0}^{n} C_p \otimes C'_{n-p}$

& $d_n : (C \otimes C')_n \longrightarrow (C \otimes C')_{n-1}$ by

$$d_n(a \otimes b) = (\partial_p a) \otimes b + (-1)^p (a \otimes \partial'_{n-p} b).$$

$$(d \circ d)(a \otimes b) = d\big((\partial a) \otimes b + (-1)^p (a \otimes \partial' b)\big)$$
$$= \underbrace{\partial^2 a}_{0} \otimes b + (-1)^{p-1} (\partial a) \otimes (\partial' b) + (-1)^p (\partial a) \otimes (\partial' b) + \underbrace{(-1)^{2p} a \otimes \partial'^2 b}_{0}$$
$$= 0$$

When we pass to homology & cohomology; homological algebra gives split s.e.s.

$$0 \to \bigoplus_p H_p(C_*) \otimes_R H_{n-p}(C'_*) \to H_n((C \otimes C')_*) \to \bigoplus \mathrm{Tor}_R(H_p(C_*), H_{n-p-1}(C'_*)) \to 0$$

Künneth formula for homology:

$$H_n(X \times Y) \cong \bigoplus_{p=0}^{n} (H_p(X) \otimes H_{n-p}(Y)) \oplus \bigoplus_{p=0}^{n-1} \mathrm{Tor}(H_p(X), H_{n-p-1}(Y))$$

Corollary: $\chi(X \times Y) = \chi(X) \cdot \chi(Y)$, $\dim(V \otimes W) = \dim V \cdot \dim W$

Künneth formula for cohomology:

$$H^n(X \times Y) \cong \bigoplus_{p=0}^{n} H^p(X) \otimes H^{n-p}(Y) \oplus \bigoplus_{p=0}^{n+1} \mathrm{Tor}(H^p(X), H^{n-p+1}(Y))$$

Exp: $H^k(S^n \times S^m; \mathbb{Q})$

= POINCARÉ DUALITY =

There is a very special symmetry