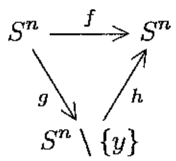
Degrees

A continuous map $f : S^n \to S^n$ gives rise to a corresponding map $f_* : \widetilde{H}_n(S^n) \to \widetilde{H}_n(S^n)$. Since $\widetilde{H}_n(S^n) \cong \mathbb{Z}$, this is a map $\mathbb{Z} \to \mathbb{Z}$, a homomorphism which is multiplication by some integer, this integer is the **degree** of the map f, denoted deg f.

Properties

1. deg $id_{S^n} = 1$
   This is because $(id_{S^n})_* = id$ which is multiplication by the integer 1.
2. If f is not surjective $\Rightarrow$ deg $f = 0$.
   Suppose f is not surjective, then there is a $y \notin \text{Im } f$. Then we can factor f in the following way:

   Since $S^n - \{y\} \cong \mathbb{R}^n$ which is contractible, $H_n(S^n \setminus \{y\}) = 0$. Therefore $f_* = h_* g_* = 0$, so deg $f = 0$.
3. If $f \cong g \Rightarrow$ deg $f =$ deg g.
   This is because $f_* = g_*$. Note that the converse is also true.
4. deg $(g \circ f) =$ deg $g \cdot$ deg f.
   Since $(g \circ f)_* = g_* \circ f_*$.
5. If f is a homotopy equivalence (there exists a g so that $g \circ f \simeq id_{S^n}$) $\Rightarrow$ deg $f = \pm 1$.
   This follows directly from 1, 3, and 4 above, since $f \circ g \cong id_{S^n}$ implies that deg $f \cdot$ deg $g =$ deg $id_{S^n} = 1$.
6. If $r : S^n \to S^n$ is a reflection across some n-dimensional subspace of $\mathbb{R}^{n+1} \Rightarrow$ deg $r = -1$. $\underline{x} = (x_0, \ldots x_n) \mapsto (-x_0, x_1, \ldots, x_n)$
   Without loss of generality we can assume the subspace is $\mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n-1}$. Choose a CW complex for S^n whose n-cells are given by Δ_1^n and Δ_2^n, the upper and lower hemispheres of S^n, attached by identifying their boundaries together in the standard way. Then consider the generator of $H_n(S^n)$: $[\Delta_1^n - \Delta_2^n]$. The reflection map r maps the cycle $\Delta_1^n - \Delta_2^n$ to $\Delta_2^n - \Delta_1^n = -(\Delta_1^n - \Delta_2^n)$. So

$$r_*([\Delta_1^n - \Delta_2^n]) = [\Delta_2^n - \Delta_1^n] = [-(\Delta_1^n - \Delta_2^n)] = -1 \cdot [\Delta_1^n - \Delta_2^n]$$

so deg $r = -1$.

7. If $a : S^n \to S^n$ is the antipodal map ($\underline{x} \mapsto -\underline{x}$) $\Rightarrow$ deg $a = (-1)^{n+1}$
Note that a is simply a composition of $n+1$ reflections, since there are $n+1$ coordinates in $\underline{x}$, each getting mapped by an individual reflection. From 4 above we know that composition of maps leads to multiplication of degrees.

8. If $f : S^n \to S^n$ and $Sf : S^{n+1} \to S^{n+1}$ is the suspension of f then deg $Sf =$ deg f.
General note about suspensions: If $f : X \to X$ and $\Sigma X = X \times [-1,1]/(X \times \{-1\}, X \times \{1\})$ (the reduced suspension of X), then $SX = f \times id_{[-1,1]}/\sim$, the same equivalence as in ΣX. Note that $\Sigma S^n = S^{n+1}$, you can see this by thinking of the equatorial part of ΣX as a copy of S^n.
The Suspension Theorem states: $\widetilde{H}_i(X) \cong \widetilde{H}_{i+1}(\Sigma X)$. By noting that $\Sigma X = C_+X \cup_X C_-X$, where C_+X and C_-X are the upper and lower cones of the suspension joined along their bases, we can prove the statement using the Mayer-Vietoris sequence:

$$\to \widetilde{H}_{n+1}(C_+X) \oplus \widetilde{H}_{n+1}(C_-X) \to \widetilde{H}_{n+1}(\Sigma X) \to \widetilde{H}_n(X) \to \widetilde{H}_n(C_+X) \oplus \widetilde{H}_n(C_-X) \to$$

Since both C_+X and C_-X are both contractible, the end groups in the above sequence are both zero. Thus by exactness $\widetilde{H}_i(X) \cong \widetilde{H}_{i+1}(\Sigma X)$ as desired.
Let C_+S^n denote the upper cone of ΣX. Note that the base of C_+S^n is $S^n \times \{0\} \subset \Sigma X$. Our map f induces a map $C_+f : (C_+S^n, S^n) \to (C_+S^n, S^n)$ whose quotient is Sf. The long exact sequence of the pair (C_+S^n, S^n) in homology gives the following commutative diagram:

$$\begin{array}{ccccccc} 0 \longrightarrow \widetilde{H}_{i+1}(C_+S^n, S^n) & \simeq & \widetilde{H}_{i+1}(C_+S^n/S^n) & \overset{\partial}{\underset{\sim}{\longrightarrow}} & \widetilde{H}_i(S^n) & \longrightarrow 0 \\ & & \downarrow{\scriptstyle (Sf)_*} & & \downarrow{\scriptstyle f_*} & \\ & & \widetilde{H}_{i+1}(S^{n+1}) & \overset{\partial}{\underset{\sim}{\longrightarrow}} & \widetilde{H}_i(S^n) & \end{array}$$

Note that $C_+S^n/S^n \cong S^{n+1}$ so the boundary map ∂ at the top and bottom of the diagram are the same map. So by the commutativity of the diagram, since f_* is defined by multiplication by some integer m, then $(Sf)_*$ is multiplication by the same integer m.

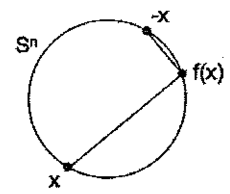
Example:

Consider the reflection map: $r_n : S^n \to S^n$ by $(x_0, \ldots, x_n) \mapsto (-x_0, x_1, \ldots, x_n)$. Since r_n leaves $x_1, x_2, \ldots, x_n$ unchanged we can unsuspend one at a time to get deg $r_n =$ deg $r_{n-1} = \cdots =$ deg r_0, where $r_i : S^i \to S^i$ by $(x_0, x_1, \ldots, x_i) \mapsto (-x_0, x_1, \ldots, x_i)$. So $r_0 : S^0 \to S^0$ by $x_0 \mapsto -x_0$. Note that S^0 is two points but in reduced homology we are only looking at one integer. Consider

$$0 \to \widetilde{H}_0(S^0) \to H_0(S^0) \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

where $\widetilde{H}_0(S^0) = \{(a,-a) \mid a \in \mathbb{Z}\}$, $H_0(S^0) = \mathbb{Z} \oplus \mathbb{Z}$, and $\epsilon : (a,b) \mapsto a+b$. Then $(r_0)_* : \widetilde{H}_0(S^0) \to \widetilde{H}_0(S^0)$ by $(a,-a) \mapsto (-a,a) = (-1)(a,-a)$.

9. If $f : S^n \to S^n$ has no fixed points then deg $f = (-1)^{n+1}$.
Consider the figure below. Since $f(x) \neq x$, the segment $(1-t)f(x) + t(-x)$ from $-x$ to $f(x)$ does not pass through the origin in $\mathbb{R}^{n+1}$ so normalize: $\frac{(1-t)f(x)+t(-x)}{|(1-t)f(x)+t(-x)|} = g_t(x) : S^n \to S^n$. Note that this homotopy is well defined since $(1-t)f(x) - tx \neq 0$ for any $x \in S^n$ and $t \in [0,1]$, because $f(x) \neq x$ for all x. Then g_t is a homotopy from f to a, the antipodal map.



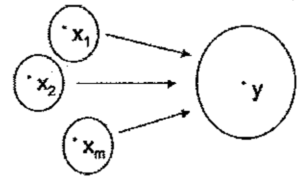
How to Compute Degrees:

Example: $f : S^1 \to S^1$, $f(x) = x^k$, $k \in \mathbb{Z}$

Claim: deg $f = k$ (*proof next time*)

Corollary: Can construct maps $f : S^n \to S^n$ of any given degree $k \in \mathbb{Z}$ by suspension. We simply take the map $f : S^1 \to S^1$ such that $f(x) = x^k$ with our desired degree k and suspend the map as many times as needed so that is now a map $S^n \to S^n$. Suspension of a map, as shown in 8 above, increases the dimension of the space on which the map is defined by one.

Assume $f : S^n \to S^n$ is surjective, and that f has the property that there exists some $y \in \text{Im}\ (S^n)$ so that $f^{-1}(y)$ is a finite number of points, so $f^{-1}(y) = \{x_1, x_2, \ldots, x_m\}$. Let U_i be a neighborhood of x_i so that all U_i's get mapped to some neighborhood V of y. So $f(U_i - x_i) \subset V - y$. We can choose the U_i to be disjoint. We can do this because f is continuous.



Let $f|_{U_i} : U_i \to V$, the restriction of f to each U_i. Then:

$$\begin{array}{ccc}
H_n(U_i, U_i - x_i) & \xrightarrow{f_*} & H_n(V, V - y) \\
\wr\text{ (excision)} & & \wr\text{ (excision)} \\
H_n(S^n, S^n - x_i) & & H_n(S^n, S^n - y) \\
\wr\text{ l.e.s.} & & \wr\text{ l.e.s.} \\
\widetilde{H}_n(S^n) & & \widetilde{H}_n(S^n) \\
\wr & & \wr \\
\mathbb{Z} & & \mathbb{Z}
\end{array}$$

The isomorphisms above from excision come from the inclusions: $(U_i, U_i - x_i) \hookrightarrow (S^n, S^n - x_i)$ inducing the isomorphism $H_n(U_i, U_i - x_i) \to H_n(S^n, S^n - x_i)$ and $(V, V\_y) \hookrightarrow (S^n, S^n - y)$ inducing the isomorphism $H_n(V, V - y) \to H_n(S^n, S^n - y)$. Note that l.e.s. is the long exact sequence of the pairs. So we have

$$\to \widetilde{H}_n(S^n - x_i) \to \widetilde{H}_n(S^n) \to H_n(S^n, S^n - x_i) \to \widetilde{H}_{n-1}(S^n - x_i) \to$$

Since $S^n - x_i$ is contractible the two relative homology groups at the ends of the above piece of the long exact sequence for the pair are both zero. So by exactness the middle two groups are isomorphic, as in the diagram above. The long exact sequence for $(S^n, S^n - y)$ yields the similar result on the right hand side of the above diagram.

Define deg $f_{x_i} :=$ effect of $f_* : H_n(U_i, U_i - x_i) \to H_n(V, V-y)$. This map f_* is multiplication by some integer.

Theorem: $\deg f = \sum_{i=1}^{m} \deg f|_{x_i}$ (the sum of local degrees) (*proof next time*).

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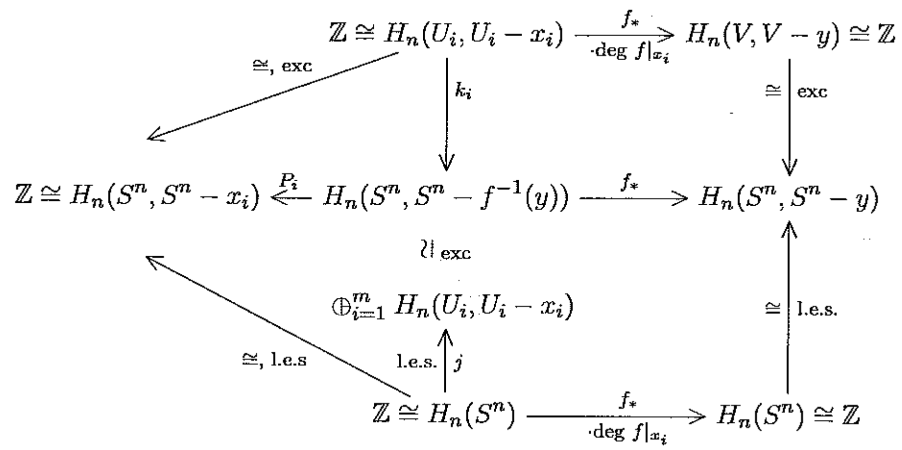
Theorem: (Continued from last time) $\deg f = \sum_{i=1}^{m} \deg f|_{x_i}$.

Proof: (Note: exc is excision)



Everything in the above diagram in commutative. Note that isomorphisms from excision and the long exact sequences were shown at the end of Monday's (1/23/12) notes. To save space, they are not repeated here. We now only need to look at k_i, P_i, and j. By examining the diagram above we have: $k_i(1) = (0, \ldots, 0, 1, 0, \ldots, 0)$ where the the 1 is in the ith place, $P_i \circ j(1) = 1$, for all i so $j(1) = (1, 1, \ldots, 1) = \sum_{i=1}^{m} k_i(1)$. The commutativity of the lower rectangle gives:

$$\deg f = f_* j(1) = f_*\left(\sum_{i=1}^{m} k_i(1)\right) = \sum_{i=1}^{m} f_*(0, \ldots, 0, 1, 0, \ldots, 0) = \sum_{i=1}^{m} \deg f|_{x_i}$$

Thus we have shown that the degree of a map f is the sum of its local degrees.

Example: From last class: $f : S^1 \to S^1$, $f(x) = x^k$, $k \in \mathbb{Z}$. **Claim:** $\deg f = k$.

- If $k = 0$ then f is the constant map which has degree 0.
- If $k < 0$ we can compose f with a reflection $r : S^1 \to S^1$ by $(x, y) \to (x, -y)$. This reflection has degree -1. So since composition leads to multiplication of degrees, we can assume that $k > 0$.
- If $k > 0$, then for all $y \in S^1$, $f^{-1}(y)$ has k points (the k roots) call them $x_1, x_2, \ldots, x_k$, and f has local degree 1 at each of these points. For our $y \in S^n$ we can find a small open neighborhood centered at y, call this neighborhood V. The pre-images of V are open neighborhoods U_i centered at each x_i. Then $f|_{U_i} : U_i \to V$ is a homeomorphism, which has possible degree ± 1. In this case the homeomorphisms are a restriction of a rotation which is homotopic to the identity and thus the degree of $f|_{U_i}$ is 1 for each i. So the degree of f is indeed k.

CW Complex:

Some notation: $X = \cup_n X_n$, where X_n (or X^n in Hatcher) is the n-skeleton which contains all cells up to and including dimension n. Then $X_n = X_{n-1} \amalg_\lambda D^n_\lambda / \sim x \in \partial D^n_\lambda \sim \varphi_\lambda(x)$, where $\partial D^n_\lambda = S^{n-1} \xrightarrow{\varphi_\lambda} X_{n-1}$. Note that φ_λ is the attaching or gluing map. So we are simply gluing the boundary of our n-cells to X_{n-1} according to our attaching map φ_λ. The CW Complex has the weak topology: $A \subset X$ is open $\iff$ $A \cap X_n$ is open for any n. Also $e^n_\lambda = \text{Int}(D^n_\lambda)$ is often how we denote an n-cell. You can think of X as a disjoint union of cells of various dimensions or as $\amalg_{n,\lambda} D^n_\lambda / \sim$, again where $\sim$ is attaching the cells via there respective attaching maps.

Example: S^n: one 0-cell (e^0) and one n-cell (e^n). The attaching map is $\phi_\alpha : S^{n-1} = \partial D^n \to$ point. There is only one such map, the collapsing map. Think of taking the disk D^n and collapsing the entire boundary to a single point, giving S^n.

Example: S^n: two cells in each dimension. $X_0 = S^0$, just two points e^0_1, e^0_2. $X_1 = S^1$ where our two 1-cells D^1_1, D^1_2 are attached to the 0-cells by homeomorphisms on the boundary. Similarly, two 2-cells can be attached to $X_1 = S^1$ by homeomorphism on the boundary giving $X_2 = S^2$. Keep working in this manner adding two cells in each new dimension. Note that if you identify each pair of cells in the same dimension with the antipodal map you get a CW structure for $\mathbb{R}P^n$.

Example: $\mathbb{R}P^n = S^n / \sim = \mathbb{R}^{n+1}/\mathbb{R}^*$, where you can think of $\mathbb{R}P^n$ as taking all lines going through the origin in which all points on the same line are identified to one point. $\mathbb{R}P^n$ has CW structure with one cell in each dimension $0, 1, \ldots, n$.

Example: $\mathbb{C}P^n = S^{2n+1}/S^1$. We can define an action: $S^1 \times S^{2n+1} \to S^{2n+1}$ by $(\lambda, (z_0, \ldots, z_n)) \mapsto (\lambda z_0, \ldots, \lambda z_n)$, the action is simply multiplication. One can also write $\mathbb{C}P^n = \{[z_0, \ldots, z_n]\}$ where $[z_0, \ldots, z_n] = (z_0 : \ldots : z_n) \sim (\lambda z_0 : \ldots : \lambda z_n)$. Also $\mathbb{C}P^n = \mathbb{C}^{n+1}/\mathbb{C}^*$ where all points on a complex line through the origin are identified to a single point. Finally $\mathbb{C}P^n \cong \mathbb{C}P^{n-1} \sqcup_\varphi D^{2n}$. Where $\psi : D^{2n} \to \mathbb{C}P^n$ by $(z_1 : \ldots : z_n) \mapsto (z_1, : \ldots : z_n : \sqrt{1 - \sum_{i=1}^n |z_i|^2})$. Then the attaching map is $\varphi = \psi|_{S^{2n-1}} : S^{2n-1} \to \mathbb{C}P^{n-1}$. It has a CW structure with one cell in each even dimension $0, 2, \ldots, 2n$.

Cellular Homology:

Lemma: If X is a CW complex, then:

(a) $H_k(X_n, X_{n-1}) = \begin{cases} 0 & \text{if } k \neq n \\ \mathbb{Z}^{\#\text{ of n-cells}} & \text{if } k = n \end{cases}$

(b) $H_k(X_n) = 0$ if $k > n$

(c) $i : X_n \hookrightarrow X$ induces an isomorphism $H_k(X_n) \to H_k(X)$ if $k < n$.

Proof of (a): Picture X_n as X_{n-1} with some number of n-cells (e^n_λ) attached to X_{n-1}. Pick a point x_λ at the center of each n-cell. Let $A = X_n - \{x_\lambda\}_\lambda$. Then A deformation retracts to X_{n-1} and we have $H_k(X_n, X_{n-1}) \cong H_k(X_n, X_n - \{X_\lambda\})$. By excision, excising X_{n-1}, this is equivalent to $H_k(D^n_\lambda, D^n_\lambda - \{x_\lambda\})$. Note that whether we consider the disks to be open or closed does not affect the homology. We consider the long exact sequence of the pair $(D^n_\lambda, D^n_\lambda - \{x_\lambda\})$:

$$\to H_n(D^n) \to H_n(D^n_\lambda, D^n_\lambda - \{x_\lambda\}) \to H_{n-1}(D^n_\lambda - \{x_\lambda\}) \to H_{n-1}(D^n) \to$$

As D^n is contractible $H_n(D^n)$ and $H_{n-1}(D^n)$ both equal zero. So by exactness the middle two groups are isomorphic. Since $D^n_\lambda - \{x_\lambda\}$ is isomorphic to S^{n-1}

we have $H_{n-1}(D^n_\lambda - \{x_\lambda\}) = H_n(S^{n-1}_\lambda)$. So we can conclude:

$$H_k(D^n_\lambda, D^n_\lambda - \{x_\lambda\}) \cong \oplus_\lambda H_{k-1}(S^{n-1}_\lambda) \cong \begin{cases} \oplus_\lambda \mathbb{Z} & \text{if } k = n \\ 0 & \text{if } k \neq n \end{cases}$$

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Continuing Proof of Lemma:

Proof of (b): Looking at the long exact sequence of the pair for (X_n, X_{n-1}) we have the following piece:

$$\to H_{k+1}(X_n, X_{n-1}) \to H_k(X_{n-1}) \to H_k(X_n) \to H_k(X_n, X_{n-1}) \to$$

If $k+1 \neq n$ and $k \neq n$, so from part (a) of the Lemma, $H_{k+1}(X_n, X_{n-1}) = 0$ and $H_k(X_n, X_{n-1}) = 0$. Thus $H_k(X_{n-1}) \cong H_k(X_n)$. Now suppose $k > n$ (so in particular, $n \neq k+1$ and $n \neq k$), thus $H_k(X_n) \cong H_k(X_{n-1}) \cong \cdots \cong H_k(X_0)$ by the corresponding long exact sequences. Note that X_0 is simply a collection of points, so $H_k(X_0) = 0$. Thus when $k > n$ we have $H_k(X_n) = 0$ as desired.

Proof of (c): First a bit of terminology: A CW complex is *finite* if it only has finitely many cells, so there is a cell of maximum dimension. A CW complex is *finite type* if it only has finitely many cells in each dimension. A CW complex of finite type could have cells in infinitely many dimensions. Finally let $X = \cup_n X_n$. If $X_m = X_n$ for all $m > n$ for some n then $X = X_n$ and we say that the skeleton *stabilizes*. Now we move on to the proof of part (c), we look only at finite dimensional CW complexes. Let $k < n$. Look at the long exact sequence for the pair (X_{n+1}, X_n):

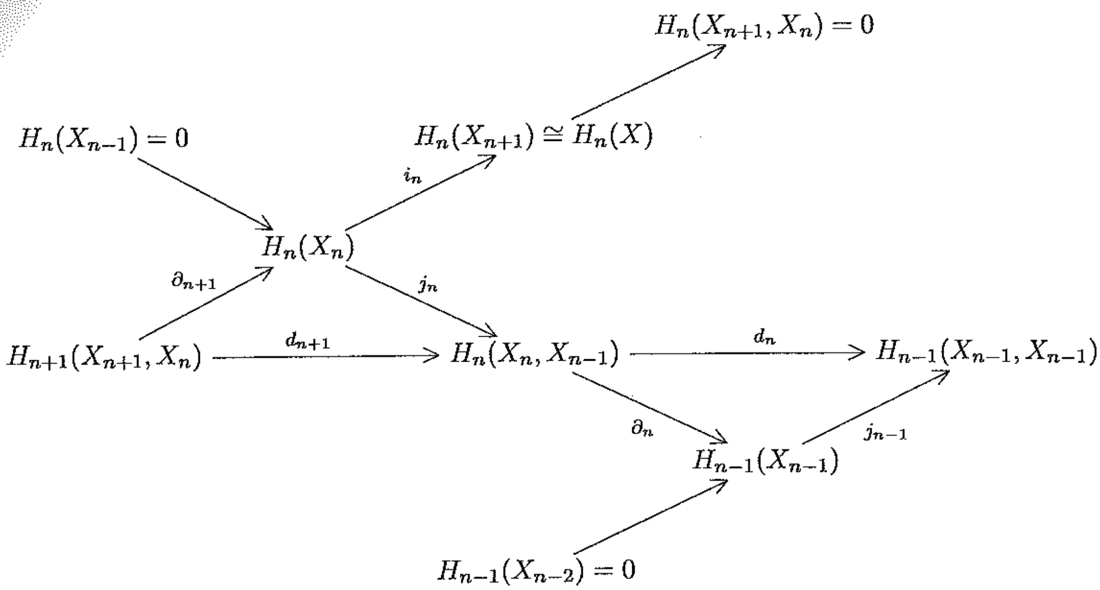
$$\to H_{k+1}(X_{n+1}, X_n) \to H_k(X_n) \to H_k(X_{n+1}) \to H_k(X_{n+1}, X_n) \to$$

Since $k < n$ we have $k+1 \neq n+1$ and $k \neq n+1$, so by part (a) $H_{k+1}(X_{n+1}, X_n) = 0$ and $H_k(X_{n+1}, X_n) = 0$. Thus $H_k(X_n) \cong H_k(X_{n+1})$. Similarly to what we did in (b) we can extend this, $H_k(X_n) \cong H_k(X_{n+1}) \cong H_k(X_{n+2}) \cong \cdots \cong H_k(X_{n+l}) = H_k(X)$. Since X is finite dimensional we know that $X = X_{n+l}$ for some l. Thus we have proved (c).

We now begin the machinery for showing that the homology of a space X with a CW structure is actually independent of the specific CW structure that we put on X. We define a chain complex indexed by the cells of X: $\mathcal{C}_n := H_n(X_n, X_{n-1}) = \mathbb{Z}^{\#n\text{-cells}}$, this follows directly from part (a) of the Lemma above. We also want to define a map $d_n : \mathcal{C}_n \to \mathcal{C}_{n-1}$ such that $d_n \circ d_{n+1} = 0$. We can define $\mathcal{C}_n$ and d_n in the way for all n. Once such a d_n is defined we let $H_i^{CW}(X) := \ker d_i / \text{Im } d_{i+1}$, this is called the Cellular Homology.

__Theorem__: $H_i^{CW}(X) \cong H_i(X)$ for all i, where $H_i(X)$ is the singular homology of X.

Proof: We need the following diagram to relate the cellular homology with singular homology:



First note that $H_n(X_{n-1}) = 0$ and $H_{n-1}(X_{n-2}) = 0$ in the diagram above by part (b) of the above Lemma. Also $H_n(X_{n+1}) \cong H_n(X)$ by part (c) of the above Lemma. We can use composition to define our map $d_n = j_{n-1} \circ \partial_n : C_n \to C_{n-1}$. We must first show that $d_n \circ d_{n+1} = 0$. To do this we use the commutativity of the above diagram to write: $d_n \circ d_{n+1} = j_{n-1} \circ \partial_n \circ j_n \circ \partial_{n+1}$. Composition is associative so we can write this as $d_n \circ d_{n+1} = j_{n-1} \circ (\partial_n \circ j_n) \circ \partial_{n+1}$. Then $(\partial_n \circ j_n) = 0$ as it is two consecutive maps in the long exact sequence down the main diagonal of the diagram above. So indeed $d_n \circ d_{n+1} = 0$. Now from the diagram above $H_n(X) \cong H_n(X_n)/\ker i_n$. By exactness, since $H_n(X_{n+1}, X_n) = 0$, we have $H_n(X) = H_n(X_n)/\ker i_n \cong H_n(X_n)/\text{Im } \partial_{n+1}$. Now, $H_n(X_n) \cong \text{Im } j_n = \ker \partial_n = \ker d_n$. The first isomorphism comes from j_n being injective. By exactness $\text{Im } j_n = \ker \partial_n$. Finally $\ker \partial_n = \ker d_n$ since $d_n = j_{n-1} \circ \partial_n$ and j_{n-1} is injective, meaning it does not affect the kernel. Also we have $\text{Im } \partial_{n+1} = \text{Im } d_{n+1}$. This is true since $d_{n+1} = j_n \circ \partial_{n+1}$ and j_n is injective. So in conclusion we have

$$H_n(X) \cong H_n(X_n)/\text{Im } \partial_{n+1} = \ker d_n/\text{Im } d_{n+1} = H_n^{CW}(X)$$

So we have proved the theorem.

Immediate Consequences:

(a) If X has no n-cells, then $H_n(X) = 0$.

By the Theorem above we have $H_n(X) = \ker d_n/\text{Im } d_{n+1}$. As X has no n-cells $H_n(X_n, X_{n-1}) = 0$ by the lemma. So $\ker d_n = 0$ and $H_n(X) = 0$.

(b) If X is connected and has a single 0-cell then $d_1 : C_1 \to C_0$ is the zero map.

Since X contains only a single 0-cell, $C_0 = \mathbb{Z}$. Since X is connected $H_0(X) = \mathbb{Z}$. By the Theorem above $H_0(X) = \ker d_0/\text{Im } d_1$. We have $\ker d_0 = \mathbb{Z}$, so $H_0(X) = \mathbb{Z}/\text{Im } d_1$. So combing everything we have $\mathbb{Z} = \mathbb{Z}/\text{Im } d_1$ which implies that $\text{Im } d_1 = 0$, so d_1 is the zero map as desired.

(c) If X has no two cells in adjacent dimensions then $d_n = 0$ for all n and $H_n(X) \cong \mathbb{Z}^{\#n-cells}$ for all n.

We show this by using Cellular Homology and looking at the sequence of chain complexes:

$$\to C_{n+1} \to C_n \to C_{n-1} \to C_{n-2} \to$$

When no two cells are in adjacent dimensions, at the very least every other C_i is zero. In this case, all the maps between the chain complexes are zero. Thus $H_n^{CW}(X) = \ker 0/\text{Im } 0 = C_n$. This is true for all n.

Example: $\mathbb{C}P^n$ has one cell in each dimension $0, 2, 4, \ldots, 2n$. So $\mathbb{C}P^n$ has no two cells in adjacent dimensions, meaning we can apply Consequence (c) above to say:

$$H_i(\mathbb{C}P^n) = \begin{cases} \mathbb{Z} & \text{if } i = 0, 2, 4, \ldots, 2n \\ 0 & \text{otherwise} \end{cases}$$

Example: When $n > 1$, $S^n \times S^n$ has one 0-cell, two n-cells, and one $2n$-cell. Since $n > 1$, these cells are not in adjacent dimensions so again Consequence (c) above applies to give:

$$H_i(S^n \times S^n) = \begin{cases} \mathbb{Z} & i = 0, 2n \\ \mathbb{Z}^2 & i = n \\ 0 & \text{otherwise} \end{cases}$$

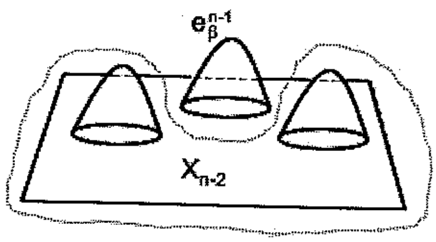
Computing the maps: $d_n : C_n = \mathbb{Z}^{\#n\text{-cells}} \to C_{n-1} = \mathbb{Z}^{\#(n-1)\text{-cells}}$

We can denote $\{e_\alpha^n\}_\alpha$ as the basis for C_n and $\{e_\beta^{n-1}\}_\beta$ as the basis for C_{n-1}. Note that e_α^n and e_β^{n-1} are considered to be open, the interior of n-disks and $(n-1)$-disks, respectively. Since our map d_n is a map $\mathbb{Z}^{\#n\text{-cells}} \to \mathbb{Z}^{\#(n-1)\text{-cells}}$ we want to be able to write: $d_n(e_\alpha^n) = \sum_\beta d_{\alpha\beta} \cdot e_\beta^{n-1}$, where $d_{\alpha,\beta} \in \mathbb{Z}$. We thus need a way of computing $d_{\alpha\beta}$.

Theorem: $d_{\alpha\beta}$ is equal to the degree of the map $S_\alpha^{n-1} \to S_\beta^{n-1}$ obtained as:

$$S_\alpha^{n-1} \xrightarrow{\varphi_\alpha} X_{n-1} = X_{n-2} \amalg_\alpha e_\gamma^{n-1} \xrightarrow{\text{collapse}} X_{n-1}/(X_{n-1} \sqcup_{\gamma \neq \beta} e_\gamma^{n-1}) = S_\beta^{n-1}$$

Note that S_α^{n-1} is obtained from the boundary of an n-cell in the basis of C_n and S_β^{n-1} is obtained from an $(n-1)$-cell in the basis of C_{n-1}. Also φ_α is the attaching map of e_α^n. The collapsing map is the map that takes all of X_{n-2} and all of the $(n-1)$-cells except the one of index β and collapses that all to a point, thus creating an S^{n-1}. This can be seen in the picture below, where the encircled part of the picture is what is collapsed down to a point.

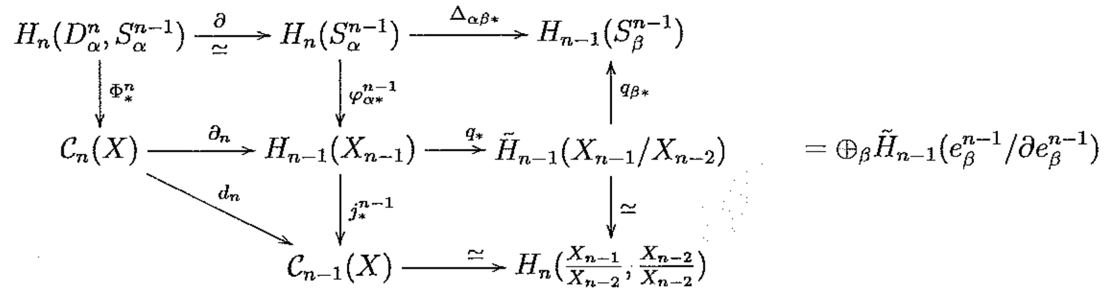


Cellular Boundary Formula

Let X be a CW complex with cells $\{e^n_\alpha\}_\alpha$ in each dimension n. Recall that if X is a CW complex we can build a cellular chain complex (C_*, d_*) where $C_* = H_n(X_n, X_{n-1}) = \mathbb{Z}^{\#\text{ of n cells}}$. To describe the cellular boundary map d_n it suffices to say what it does to the generators of the chain group C_n. To do this, we choose the natural basis for C_n which assigns to each e^n_α a generator of C_n and to each e^{n-1}_β a generator of C_{n-1}. It follows from linearity that $d_n(e^n_\alpha) = \sum_\beta d_{\alpha\beta} e^{n-1}_\beta$ for some integers $d_{\alpha\beta}$. Our first theorem will give a convenient representation to the coefficients $d_{\alpha\beta}$.

Theorem. *The coefficients $d_{\alpha\beta}$ in $d_n(e^n_\alpha) = \sum_\beta d_{\alpha\beta} e^{n-1}_\beta$ may be computed as the degree of the map $S^{n-1}_\alpha \xrightarrow{\varphi^n_\alpha} X_{n-1} \longrightarrow S^{n-1}_\beta$ given by attaching the cell e^n_α to X_{n-1} via the attaching map φ^n_α from the CW structure on X and then collapsing $X_{n-2} \coprod_{\gamma \neq \beta} e^{n-1}_\gamma$ to a point.*

Proof. We will proceed with the proof by chasing the following diagram, where the map $\Delta_{\alpha\beta *}$ is defined so that the top right square commutes.



Recall that our goal is to compute $d_n(e^n_\alpha)$. The upper left square is natural and therefore commutes (it is induced by the map $\Phi : (D^*, S^{*-1}) \to (X_*, X_{*-1})$, see Hatcher p.127), while the lower left triangle is part of the exact diagram defining the chain complex $C_*(X)$ and is defined to commute as well. Appealing to naturality, the map Φ gives a unique D^n_α so that $\Phi^n(D^n_\alpha) = e^n_\alpha$. Since the top left square and the bottom left triangle both commute, this gives that $d_n(e^n_\alpha) = j^{n-1}_* \circ \varphi^n_{\alpha *} \circ \partial(D^n_\alpha)$.

Looking to the bottom right square, recall that since X is a CW complex, (X_n, X_{n-1}) is a good pair. This gives the isomorphism $C_{n-1}(X) = H_{n-1}(X_{n-1}, X_{n-2}) \simeq \tilde{H}_{n-1}(X_{n-1}/X_{n-2})$ But, we similarly have $\tilde{H}_{n-1}(X_{n-1}/X_{n-2}) \simeq H_{n-1}(X_{n-1}/X_{n-2}, X_{n-2}/X_{n-2})$, where the isomorphism is induced by the quotient map q collapsing X_{n-2} (Hatcher 2.22 p.124).

The bottom right square commutes by the definition of j^{n-1}_* and q_* from which it follows that $d_n(e^\alpha_n) = q_* \varphi^{n-1}_{\alpha *} \partial D^\alpha_n$ where formally we should precompose with the isomorphism between $C_{n-1}(X)$ and $\tilde{H}_{n-1}(X_{n-1}/X_{n-2})$ in the left hand side so that everything is in the same space. This last map takes the generator D^α_n to some linear combination of generators in $\oplus_\beta \tilde{H}_{n-1}(e^{n-1}_\beta / \partial e^{n-1}_\beta)$. To see which generators it maps to, we project down to the β summands to obtain $d_n(e_n^\alpha) = \sum_\beta q_{\beta*}q_*\varphi_{\alpha*}^{n-1}\partial D_n^\alpha$. As noted before, we have defined $\Delta_{\alpha\beta*} = q_{\beta*}q_*\varphi_{\alpha*}^{n-1}$. Writing $d_n(e_n^\alpha) = \sum_\beta \Delta_{\alpha\beta*}\partial D_n^\beta$, we can see from the definition of the above maps that $\Delta_{\alpha\beta*}$ is multiplication by $d_{\alpha\beta}$. □

Computing Cellular Homology

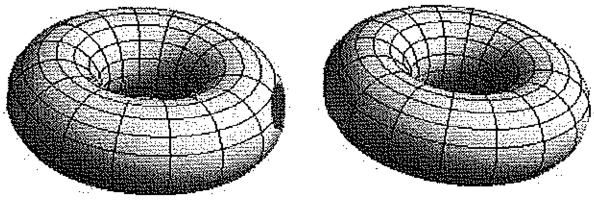
Example. M_g (the orientable surface of genus g)

We first claim that we can construct a CW structure on M_g as follows:

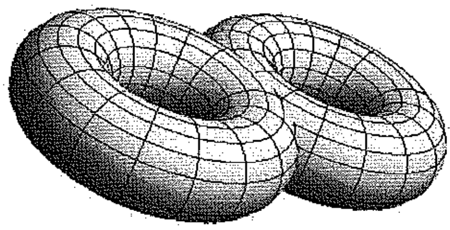
$$M_g = \{D^0\}\coprod_{\alpha=1}^{2g}\{D_\alpha^1\}\coprod\{D^2\}/[\varphi_\alpha^1, \varphi^2]$$

where we denote by $[\varphi_\alpha^1, \varphi^2]$ the relations $\varphi_\alpha^n(x) \sim x$ for $x \in \partial D_\alpha^n$ and where the maps φ_α^1 are the collapsing maps and the map φ^2 takes the generator of D^2 to the word $a_1b_1a_1^{-1}b_1^{-1}...a_gb_ga_g^{-1}b_g^{-1}$ if we write a_g and b_g as the generators associated to the $2g$ copies of D^1. We can justify that this definition agrees with the geometric definition of M_g by writing M_g as the g-fold connected sum of the torus T^2 and considering the fundamental polygon of T^2.

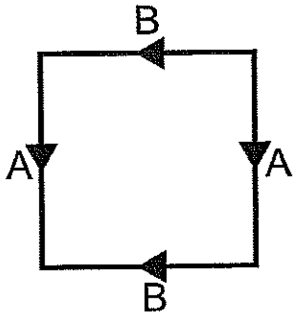
As an example of this construction in the case $g = 2$ we can consider two tori as below, where we remove the two red circles and identify their boundaries:



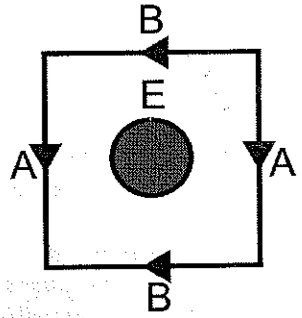
Topologically, once we glue the tori together along the identified circles, this gives the double torus (M_2) seen below:



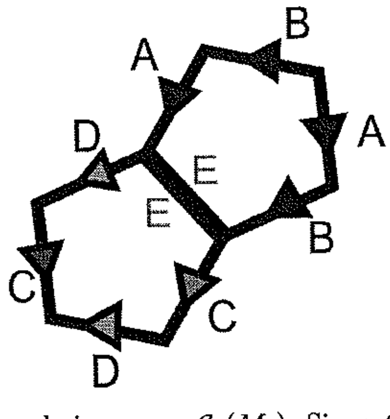
We can realize this construction at the level of the fundmanetal polygons (which will give rise more naturally to the cell structure above) by considering the fundamental polygon of the torus:



The above polygon has a natural CW structure which is exactly what we described above (the polygon above can be viewed as a single vertex identified as the boundary of each of the four edges attached to a two cell in the center where the attaching map is consistent with the identifications of the edges). Taking two copies of this fundamental polygon (one for AB and one for CD), we can remove a disk from each and identify the boundary circles E to construct the connected sum.



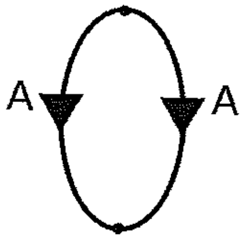
If we cut along a diagonal (the picture below corresponds to cutting the AB square from the bottom left vertex to E and the CD square from the top right vertex to E then gluing them) we can glue two of these squares together to obtain a space topologically equivalent to the above construction. Notice that we may delete all of E except its endpoints in this new picture since it is topologically the same as the area around it.



We now compute the cellular chain group $\mathcal{C}_*(M_g)$. Since $\mathcal{C}_k(M_g) \simeq H_n(X_n, X_{n-1})$ where X_n is the $n-$skeleton of M_g we have

$$0 \xrightarrow{d_3} \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^{2g} \xrightarrow{d_1} \mathbb{Z} \xrightarrow{d_0} 0$$

We demonstrated in the last class that d_1 is the zero map because there is only one one cell. There is only one two-cell as well, so we only need to find $d_2(e^2)$ which we compute via the cellular boundary formula above. Recall from the beginning of this example that the attaching map sends the generator to $a_1b_1a_1^{-1}b_1^{-1}...a_gb_ga_g^{-1}b_g^{-1}$. Consider the summand d_{ea_1} in $\sum_\beta d_{e\beta}e^1_\beta$. When we collapse all the other cells to a point, the word defining the attaching map $a_1b_1a_1^{-1}b_1^{-1}...a_gb_ga_g^{-1}b_g^{-1}$ reduces to $a_1a_1^{-1}=0$. We can see this geometrically in the case $g=2$ in the above picture by collapsing all the cells except A to a point, at which point the map is given by



By symmetry then it follows that $d_2(e^2)$ is the zero map. It then follows that the homology groups of M_g are given by

$$H_n(M_g)=\begin{cases}\mathbb{Z} & \text{i=0,2}\\ \mathbb{Z}^{2g} & \text{i=1}\\ 0 & \text{otherwise}\end{cases}$$

Example. N_g (the nonorientable surface of genus g)
Formally, $N_g=\mathbb{RP}^2\#...\#\mathbb{RP}^2$. We construct this directly as above via the labelling scheme which sends a generator to $a_1^2...a_g^2$ which coincides in dimension two ($g=1$) with $\mathbb{RP}^2$.

The short exact sequence for the cellular chains is now given by

$$0\xrightarrow{d_3}\mathbb{Z}\xrightarrow{d_2}\mathbb{Z}^g\xrightarrow{d_1}\mathbb{Z}\xrightarrow{d_0}0$$

where as before $d_1=0$ because there is only one cell in dimension one. Now to compute $d_2(e^2)$ we again apply the cellular boundary formula. The same collapsing happens in the quotient space except now $d_{ea_i}=2$ for each i since we attach along the map $\prod_i a_i^2$. It follows then that $d_2(e)=(2,...,2)$. We now choose a basis for $\mathbb{Z}^g$ given by $(1,0,...,0),...,(0,...,0,1,0),(1,...,1)$ (that is the standard basis with the last basis vector replaced with $(1,...,1)$). We can then compute directly that

$$H_n(N_g)=\begin{cases}\mathbb{Z} & \text{i=0}\\ \mathbb{Z}^{g-1}\oplus\mathbb{Z}_2 & \text{i=1}\\ 0 & \text{otherwise}\end{cases}$$

Example. $\mathbb{RP}^n$

We showed in the last notes that we can construct $\mathbb{RP}^n$ as the quotient space of S^n where we identify antipodal points. $\mathbb{RP}^n=S^n/\sim$ has one cell in each dimension $0,1,...,n$ where we consider the CW structure of S^n with two cells in each dimension. The attaching map of e^k in $\mathbb{RP}^n$ is the two-fold cover $S^{k-1}\to\mathbb{RP}^{k-1}$.

We now consider the long exact sequence for the chain groups

$$0\xrightarrow{d_{n+1}}\mathbb{Z}\xrightarrow{d_n}...\mathbb{Z}\xrightarrow{d_1}\mathbb{Z}\xrightarrow{d_0}0$$

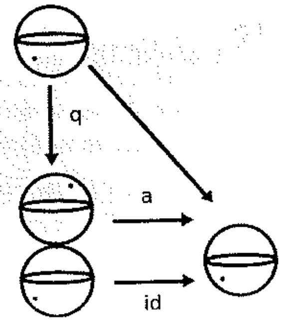
Since d_k is a map from $d_k : S^{k-1} \to S^{k-1} = \mathbb{RP}^{k-1}/\mathbb{RP}^{k-2}$ we can compute its degree directly by considering any $y \in \mathbb{RP}^{k-1} \backslash \mathbb{RP}^{k-2}$ for which we know that (letting φ be the attaching map) $\varphi^{-1}(y) = \{y, a \circ y\}$ where a is the antipodal map. This is the usual two-fold cover of S^{k-1} and so is a local homeomorphism. We consider a neighborhood V of y and the two neighborhoods U_1 and U_2 given to exist by the local homeomorphism property. Then by the local degree formula proven in the previous notes $d_k = \deg \varphi = \deg id + \deg id \circ a = 1 + (-1)^k$. It follows that

$$d_k = \begin{cases} 0 & \text{if k is odd} \\ 2 & \text{if k is even} \end{cases}$$

and therefore we obtain that

$$H_k(\mathbb{RP}^n) = \begin{cases} \mathbb{Z}_2 & \text{if k is odd } k < n \\ \mathbb{Z} & k = 0, n \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

A different approach to this problem would be to invert the order in which we collapse $\mathbb{RP}^{n-1}$ and identify points as in the diagram below, where q is the map collapsing S^{k-2} to a point, id is the identity map, and a is the antipodal points map.



Once we unravel the definitions it is immediate that the computations here are the same as above (formally this follows from the fact that $\deg fg = \deg f \deg g = \deg g \deg f$).

Euler Characteristic

We will now define an invariant related to cellular homology which, when generalized a bit, will lead to a generalization of the Brouwer fixed point theorem.

Definition. *Let X be a finite CW complex of dimension n and denote by c_i the number of i cells of X. Then $\chi(X) = \sum_{i=0}^{n} c_i$ is defined to be the Euler characteristic of X.*

Given such a definition, it is natural to question whether or not it depends on the cell structure chosen for the space X. This is not the case. We established previously that cellular homology is isomorphic to singular homology. In particular then since singular homology is indpendent of the cell structure on X it suffices to show that the Euler characteristic depends only on the cellular homology of the space X.

Theorem. $\chi(X) = \sum_{i=0}^{n}(-1)^i$ *rank*$(H_i(X))$ *where the rank of a group is its Betti number. In particular, $\chi(X)$ is independent of cell structure.*

Proof. We will follow the notation in Hatcher. Call $B_n = \text{Im } d_{n+1}$, $Z_n = \text{Ker } d_n$, and $H_n = Z_n/B_n$ and consider the long exact sequence of chain complexes and the short exact sequences defining homology:

$$0 \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} \cdots \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{d_0} 0$$

$$0 \longrightarrow Z_n \overset{\iota}{\hookrightarrow} C_n \overset{d_n}{\twoheadrightarrow} B_{n-1} \longrightarrow 0$$

$$0 \longrightarrow B_n \xrightarrow{d_{n+1}} Z_n \xrightarrow{q} H_n \longrightarrow 0$$

The rank nullity theorem implies rank $C_n =$ rank $Z_n +$ rank B_n and rank $Z_n =$ rank $B_n +$ rank H_n. It then follows that $\chi(X) = \sum_{i=0}^{n}(-1)^i \text{rank } H_i$. □

Example. Returning to the examples above, we might consider the orientable and nonorientable surfaces of genus g. From our previous computations we have $\chi(M_g) = 1 - 2g + 1 = 2(1-g)$ and $\chi(N_g) = 1 - g + 1 = 2 - g$ so that $\chi(M_g) = \chi(N_{2g})$.

Lefschetz Fixed Point Theorem

Recall that if G is a finitely generated abelian group, then $G \simeq \mathbb{Z}^r \oplus_{i=1}^n \mathbb{Z}_{p_i^{r_i}}$. As in the previous proof, it will be convenient to ignore the effect of the torsion part of maps. Given an endomorphism $\varphi : G \to G$ define Tr $(\varphi) =$ Tr $(\bar{\varphi} : G/\oplus_{i=1}^n \mathbb{Z}_{p_i^{r_i}} \to G/\oplus_{i=1}^n \mathbb{Z}_{p_i^{r_i}})$ where the latter trace is the linear algebraic trace of the maps $\bar{\varphi} : \mathbb{Z}^r \to \mathbb{Z}^r$.

Definition. *If X has the homotopy type of a finite simplicial or cellular complex and $f : X \to X$, then the Lefschetz number of f is defined to be $\tau(f) = \sum_i (-1)^i Tr\ (f_* : H_i(X) \to H_i(X))$.*

Notice that homotopic maps have the same Lefschetz number since they induce the same maps on homology.

Example. Suppose that $f \simeq id_X$, then $\tau(f) = \chi(X)$. This follows from the fact the map induced by the identity is the identity matrix and that the trace of the identity matrix in this case will be the relevant Betti number of X.

Theorem. *(Lefschetz) If X is a retract of a finite simplicial complex and if $f : X \to X$ satisfies $\tau(f) \neq 0$ then f has a fixed point.*

Before proving this result, we consider a few examples, then look to an intermediate result, the simplicial approximation theorem.

Example. Suppose that X has the homology of a point (up to torsion), then $\tau(f) =$ Tr $(f_* : H_0 \to H_0) = 1$. This computation follows from the fact that all the other homology groups are zero and that the map induced on H_0 is the identity.

This leads immediately to two nontrivial results, the first of which is the Brouwer fixed point theorem.

Example. (Brouwer) If $f : D^n \to D^n$ is continuous then f has a fixed point. This is an immediate consequence of the above result.

Example. If $X = \mathbb{RP}^{2n}$ then modulo torsion we computed above that X has the homology of a point and therefore any map $f : \mathbb{RP}^{2n} \to \mathbb{RP}^{2n}$ has a fixed point.

Finally we are led to an example which does not follow from the computation for a point.

Example. If $f : S^n \to S^n$ and deg $f \neq (-1)^{n+1}$ then f has a fixed point. To verify this, we compute $\tau(f) = \text{Tr } (f_* : H_0(S^n) \to H_0(S^n)) + (-1)^n \text{Tr } (f_* : H_n(S^n) \to H_n(S^n)) = 1 + (-1)^n \deg f$.

Corollary. *If $a : S^n \to S^n$ is the antipodal map, then deg $a = (-1)^{n+1}$.*

Now we return to outlining the proof:

Definition. *If K and L are simplicial complexes and $f : K \to L$ is a linear map which sends each simplex of K to a simplex in L so that vertices map to vertices, then f is said to be simplicial.*

The simplicial approximation theorem says that that given any map f from a finite simplicial complex to an arbitrary simplicial complex, we can find a map g in the homotopy class of f so that g is simplicial in the above sense with respect to some finite iteration of barycentric subdivisions of the domain.

Theorem. *If K is a finite simplicial complex and L is an arbitrary simplicial complex, then for any map $f : K \to L$ there is a map in the homotopy class of f which is simplicial with respect to a finite interation of barycentric subdivisions of K.*

The proof of this result is omitted. We now proceed to the Lefschetz theorem.

Proof. (sketch of proof) We proceed by showing the contrapositive, so suppose that f has no fixed points. The general case reduces to the case that X is a finite simplicial complex, since the function will have the same fixed points when we compose with the retract and because the retract induces a projection onto one of the direct summands in the homology group. We therefore take X to be a finite simplicial complex K.

K is compact and there exists a metric d on X so that d restricts to the Euclidean metric on each simplex of X; take such a metric. If f has no fixed points, we can find a uniform ϵ for which $d(x, f(x)) > \epsilon$ by the standard covering trick. Via repeated barycentric subdivision of X we can construct L so that for each vertex, the union of all simplicies containing that vertex has diameter less than $\frac{\epsilon}{2}$. Applying the simplicial approximation theorem we can find a subdivision K of L and a simplicial map $g : K \to L$ so that g lies in the homotopy class of f. Moreover, we may take g so that $f(\sigma)$ lies in the subcomplex of X consisting of all simplicies containing σ. Again, by repeated barycentric subdivision we may choose K so that each simplex in K has diameter less than $\frac{\epsilon}{2}$. In particular then $g(\sigma) \cap \sigma = \emptyset$ for each $\sigma \in K$. Notice $\tau(g) = \tau(f)$ since f and g are homotopic.

Since g is simplicial K^n maps to L^n (that is, it sends n skeletons to n skeletons). We constructed K as a subdivision of L so that $g(K^n) \subset K^n$ for each n.

We will use the algebraic fact that trace is additive for short exact sequences to show that we can replace $H_i(X)$ with $H_i(K^i, K^{i-1})$ in our computation of the Lefschetz number. By essentially the same argument as was used above in the computation of the Euler characteristic and using this fact we obtain that

$$\tau(g) = \sum_i (-1)^i \text{Tr } (g_* : H_i(K^i, K^{i-1}) \to H_i(K^i, K^{i-1}))$$

We have a natural basis for $H_i(K^i, K^{i-1})$ coming from the simplicies σ^i in K^i. But since $g(\sigma) \cap \sigma = \emptyset$ it follows that $\text{Tr}(g_* : H_i(K^i, K^{i-1}) \to H_i(K^i, K^{i-1})) = 0$ for each i.

□

Homology with General Coefficients

Let G be an Abelian group and X a topological space. We construct a homology with G coefficients ($H_i(X;G)$) as the homology of the chain complex

$$C_i(X;G) = C_i(X) \otimes G = \{ \sum_{i \text{ finite}} \eta_i \sigma_i, \sigma : \Delta_i \to X, \eta_i \in G\}$$

with boundary map given by $\partial_i^G = \partial_i \otimes id_g$. Since ∂_i satisfies $\partial_i \circ \partial_{i+1} = 0$ it follows that $\partial_i^G \circ \partial_{i+1}^G = 0$. We can construct versions of the usual modified homology groups (relative, reduced, etc.) in the natural way. Define relative homology by $H_i(X, A; G) = C_i(X;G)/C_i(A;G)$ and reduced homology via the augmented chain complex

$$\cdots \xrightarrow{\partial_{i+1}^G} C_i(X;G) \xrightarrow{\partial_i^G} \cdots \xrightarrow{\partial_2^G} C_1(X;G) \xrightarrow{\partial_1^G} C_0(X;G) \xrightarrow{\epsilon} G \longrightarrow 0$$

where $\epsilon(\sum \eta_i \sigma_i) = \sum_i \eta_i$. Finally, we can build cellular homology in the same way, defining $C_i^G(X) = H_i(X, i, X_{i-1}; G)$. Notice that $H_i(X) = H_i(X, \mathbb{Z})$ by definition.

By looking directly at the chain maps as before, it follows that $H_i(pt; G) = G$ if $i = 0$ and 0 otherwise. Nothing (other than coefficients) needs to change in our previous proofs about the relationships between relative homology and reduced homology of quotient spaces so we can compute the homology of the sphere as before by induction and using the long exact sequence of the pair (D^n, S^n) to be $H_i^G(S^n) = G$ if $i = 0, n$ and 0 otherwise.

Example. We compute $H_i(\mathbb{RP}^n; \mathbb{Z}_2)$ using the calculation above. Notice that over $\mathbb{Z}$ the cellular boundary maps are $d_i = 0$ or $d_i = 2$ depending on the parity of i and therefore in $\mathbb{Z}_2$ all of the boundary maps are the zero map. Then $H_i(\mathbb{RP}^n; \mathbb{Z}_2) = \mathbb{Z}_2$ for $i \leq n$ and 0 otherwise.

We now consider the question of whether it is possible for a map which is not homotopic to a constant to induce the homology of the constant map when viewed over $\mathbb{Z}$.

Example. Fix $n > 0$ and let $g : S^n \to S^n$ be a map of degree m. Define a CW complex $X = S^n \cup_g e^{n+1}$, where the notation $\cup_g$ means that we attach ∂e^{n+1} to S^n via the map $z \to g(z)$ and let f be the quotient map $f : X \to X/S^n$. Define $Y = X/S^n = S^{n+1}$. We begin by computing the usual cellular homology of X by considering the chain complex

$$0 \xrightarrow{d_{n+2}} \mathbb{Z} \xrightarrow[m]{d_{n+1}} \mathbb{Z} \xrightarrow{d_n} \cdots \xrightarrow{d_1} 0 \xrightarrow{d_1} \mathbb{Z} \xrightarrow{d_0} 0$$

and therefore

$$H_i(X;\mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}_m & i = n \\ 0 & \text{otherwise} \end{cases}$$

Moreover as $Y = S^{n+1}$ we have

$$H_i(Y;\mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0, n+1 \\ 0 & \text{otherwise} \end{cases}$$

We now consider the chain complex induced by f

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & C_{n+1}(X,\mathbb{Z}) & \xrightarrow{d_{n+1}} & C_n(X,\mathbb{Z}) & \xrightarrow{d_n} & C_{n-1}(X,\mathbb{Z}) & \xrightarrow{d_{n-1}} & \cdots \\
& \downarrow f_{\#} & & \downarrow f_{\#} & & \downarrow f_{\#} & & \\
\cdots \longrightarrow & C_{n+1}(Y,\mathbb{Z}) & \xrightarrow{d_{n+1}} & C_n(Y,\mathbb{Z}) & \xrightarrow{d_n} & C_{n-1}(Y,\mathbb{Z}) & \xrightarrow{d_{n-1}} & \cdots
\end{array}$$

which induces a homomorphism $f_* : H_*(X;\mathbb{Z}) \to H_*(Y;\mathbb{Z})$. But $H_{n+1}(X;\mathbb{Z})$ is trivial and therefore $f_{\#}$ must induce the trivial homomorphism in dimension $n+1$. Similarly, $H_n(Y,\mathbb{Z})$ is trivial and so $f_{\#}$ must be the trivial homomorphism in dimension n. All other maps are trivial by definition and so $f_{\#}$ induces the trivial homomorphism when viewed with $\mathbb{Z}$ coefficients.

We consider now $H_*(X;\mathbb{Z}_m)$ where m is, as above, the degree of the map g. We return to the chain complex level and observe that we have

$$0 \xrightarrow{d_{n+2}} \mathbb{Z}_m \xrightarrow[m]{d_{n+1}} \mathbb{Z}_m \xrightarrow{d_n} \cdots \xrightarrow{d_1} 0 \xrightarrow{d_1} \mathbb{Z}_m \xrightarrow{d_0} 0$$

Multiplication by m is now the zero map, and so we have

$$H_i(X;\mathbb{Z}_m) = \begin{cases} \mathbb{Z}_m & i = 0, n, n+1 \\ 0 & \text{otherwise} \end{cases}$$

and from the computation preceeding this example we have

$$H_i(Y;\mathbb{Z}_m) = \begin{cases} \mathbb{Z}_m & i = 0, n+1 \\ 0 & \text{otherwise} \end{cases}$$

Returning to the chain level, we will show that the induced map $f_* : H_{n+1}(X;\mathbb{Z}_m) \to H_{n+1}(X;\mathbb{Z}_m)$ is in fact injective and thus not homotopic to the constant map.

As noted before, we still have an isomorphism $\tilde{H}_{n+1}(Y;\mathbb{Z}_m) \simeq H_{n+1}(X, S^n;\mathbb{Z}_m)$. This leads us to consider the behavior of the long exact sequence of the pair (X, S^n) in dimension $n+1$. We have

$$\cdots \longrightarrow H_{n+1}(S^n;\mathbb{Z}_m) \longrightarrow H_{n+1}(X;\mathbb{Z}_m) \xrightarrow{f_*} H_{n+1}(X, S^n;\mathbb{Z}_m) \longrightarrow \cdots$$

But, $H_{n+1}(S^n;\mathbb{Z}_m) = 0$ and so f_* is injective. Since $H_{n+1}(X;\mathbb{Z}_m) = \mathbb{Z}_m \neq 0$ and $H_{n+1}(X, S^n;\mathbb{Z}_m) \simeq \tilde{H}_{n+1}(Y;\mathbb{Z}_m)$ it follows that f_* is not trivial on H_{n+1}.

Algebraic Topology Notes Week 3

π_1 vs. H_1

A continuous map $f : I \to X$ can be considered both as a path in X and a singular 1-simplex. A loop $f : I \to X$, $(f(0) = f(1))$, when regarded as a 1-simplex satisfies: $\partial f = f(1) - f(0) = 0$ so f is in fact a 1-cycle in X.

<u>**Theorem**</u> Regarding loops in a topological space X as 1-cycles, we get a homomorphism $\pi_1(X, x_0) \xrightarrow{h} H_1(X)$. If X is path-connected, h is surjective and $ker\ h = [\pi_1(X, x_0), \pi_1(X, x_0)]$. So for path-connected X:

$$\frac{\pi_1(X, x_0)}{[\pi_1(X, x_0), \pi_1(X, x_0)]} = \pi_1^{ab}(X, x_0) \cong H_1(X).$$

<u>Example</u>. Let K be the Klein Bottle ($K = \mathbb{RP}^2 \# \mathbb{RP}^2$). Now $H_2(K) = 0$ since K is non-orientable, and $H_0(K) = \mathbb{Z}$ since K is connected.

$$\pi_1(K) = \langle a, b | aba^{-1}b \rangle$$
$$\pi_1^{ab}(K) = \frac{\langle a \rangle \oplus \langle b \rangle}{\langle 2b \rangle} \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}.$$

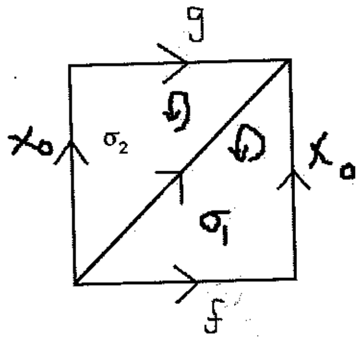
<u>Remark</u>. Equivalent definition of h: For $[f : S^1 \to X] \in \pi_1(X)$, h sends $[f] \mapsto f_*(gen) \in H_1(X)$ where gen is a generator of $H_1(S^1)$. In particular, we can take $\sigma : I \to S^1$ such that $t \mapsto e^{2\pi i t}$. Then both $[f] \in \pi_1(X)$ and $f_*(gen)$ are represented by $I \xrightarrow{\sigma} S^1 \xrightarrow{f} X$. Since $f \simeq g$ yields that $f_* = g_*$ in H_1, this shows that h is well-defined.

Proof of Theorem: First we establish some notation used throughout the proof, $f \sim_p g$ denotes the existence of a basepoint preserving path-homotopy, that is $[f] = [g] \in \pi_1(X, x_0)$, and $f \sim_h g$ denotes homologous 1-cycles, that is $[f] = [h] \in H_1(X)$. The proof is split into many steps. The first four steps establish that $h : \pi_1(X, x_0) \to H_1(X)$ is a well-defined homomorphism.

Step (i): The map h preserves identity. If $f = c_{x_0}$ the constant path at x_0, we want to show $f = \partial\sigma$ for some σ a 2-simplex. Now f factors $f : I \to \{x_0\} \hookrightarrow X$. Define $\sigma : \Delta^2 = [v_0, v_1, v_2] \to X$ by $\sigma \equiv x_0$. Then

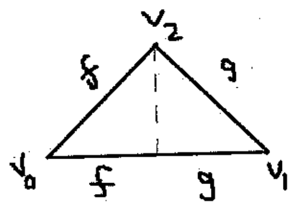
$$\begin{aligned} \partial\sigma &= \sigma|_{[v_1,v_2]} - \sigma|_{[v_0,v_2]} + \sigma|_{[v_0,v_1]} \\ &= c_{x_0} - c_{x_0} + c_{x_0} = c_{x_0} = f. \end{aligned}$$

Step (ii): The map h is well-defined as a map of sets. We need to show that if $f \sim_p g$ then $f \sim_h g$. Take a basepoint preserving homotopy $F : I \times I \to X$ from f to g. Then we can split F into two 2-simplices



Then $F_t(0) = f(0) = g(0) = x_0$, $F_t(1) = f(1) = g(1) = x_0$, and $\partial(\sigma_1 - \sigma_2) = f - g + c_{x_0} - c_{x_0} = f - g$. Now since $f - g$ is a boundary, $f \sim_h g$.

Step (iii): The map h preserves group operations. We need to show that if $f, g \in \pi_1(X, x_0)$, then $f * g \sim_h f + g$, where $f * g$ is the product path of f and g. Let $\sigma : [v_0, v_1, v_2] \to X$ be the composition $[v_0, v_1, v_2] \xrightarrow{p} [v_0, v_1] \xrightarrow{f*g} X$, where p is the projection shown below.



Now $\partial\sigma = f - f * g + g$ and thus, $f * g \sim_h f + g$.

Step (iv): The map h preserves inverses. We want to show that if $\bar{f}$ is the inverse path to f, then $\bar{f} \sim_h -f$. By (iii) $f + \bar{f} \sim_h f * \bar{f} \sim_p c_{x_0}$ and by (i) $c_{x_0} \sim_h 0$ and thus, $f + \bar{f} \sim_h 0$.

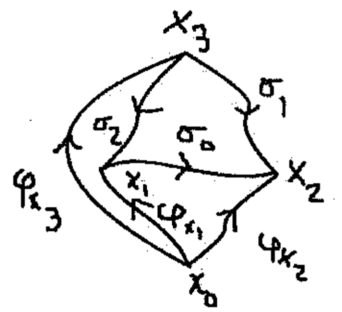
Now (i) - (iv) show that h is a well-defined homomorphism $\pi_1(X, x_0) \xrightarrow{h} H_1(X)$. Since $H_1(X)$ is abelian, there is an induced homomorphism

$$h : \pi_1^{ab}(X, x_0) = \frac{\pi_1(X, x_0)}{[\pi_1(X, x_0), \pi_1(X, x_0)]} \to H_1(X).$$

Let X be a path connected space. To prove the theorem, we construct $j : H_1(X) \to \pi_1^{ab}(X, x_0)$ and show that $h \circ j = id$ and $j \circ h = id$.

Let σ be a 1-chain in X, say $\sigma(0) = x_1$ and $\sigma(1) = x_2$. For any $x \in X$, fix a path φ_x from x_0 to x. Let $\hat{\sigma} := \varphi_{x_1} * \sigma * \overline{\varphi_{x_2}} \in \pi_1(X, x_0)$. Let $k : C_1(X) \to \pi_1(X, x_0)$ be defined by $\sigma \mapsto \hat{\sigma}$. Define $j : Z_1(X) \to \pi_1^{ab}(X, x_0)$ to be the composition $Z_1(X) \hookrightarrow C_1(X) \xrightarrow{k} \pi_1(X, x_0) \twoheadrightarrow \pi_1^{ab}(X, x_0)$. For j to be a well defined map $H_1(X) \to \pi_1^{ab}(X, x_0)$, we must show that $B_1(X)$ maps to the commutator subgroup of $\pi_1(X)$. In fact we show that $B_1(X) \mapsto c_{x_0}$.

If ρ is a 2-simplex, then $\partial\rho = \sigma_0 - \sigma_1 + \sigma_2$ and k maps $\partial\rho$ to



$$\varphi_{x_1} * \sigma_0 * \overline{\varphi_{x_2}} * \varphi_{x_2} * \overline{\sigma_1} * \overline{\varphi_{x_3}} * \varphi_{x_3} * \sigma_2 * \overline{\varphi_{x_1}}.$$

Now by associativity of the path product, this is path homotopic to

$$\varphi_{x_1} * \sigma_0 * (\overline{\varphi_{x_2}} * \varphi_{x_2}) * \overline{\sigma_1} * (\overline{\varphi_{x_3}} * \varphi_{x_3}) * \sigma_2 * \overline{\varphi_{x_1}}$$
$$\varphi_{x_1} * \sigma_0 * (c_{x_0}) * \overline{\sigma_1} * (c_{x_0}) * \sigma_2 * \overline{\varphi_{x_1}}$$
$$\varphi_{x_1} * \sigma_0 * \overline{\sigma_1} * \sigma_2 * \overline{\varphi_{x_1}}.$$

We now wish to show that $\sigma_0 * \overline{\sigma_1} * \sigma_2 \sim_p c_{x_0}$. But Δ^2 is simply connected and $\rho : \Delta^2 \to X$ so $\rho_*(\partial\rho) = \sigma_0 * \overline{\sigma_1} * \sigma_2 = c_{x_0}$ and thus, $\sigma_0 * \overline{\sigma_1} * \sigma_2 \sim_p c_{x_0}$. It follows that $\varphi_{x_1} * \sigma_0 * \overline{\sigma_1} * \sigma_2 * \overline{\varphi_{x_1}} \sim_p \varphi_{x_1} * \overline{\varphi_{x_1}} \sim_p c_{x_0}$.

Thus, the induced map $j : H_1(X) = Z_1(X)/B_1(X) \to \pi_1^{ab}(X, x_0)$ is a homomorphism.

We wish to show that $j \circ h = id$. If $\sigma \in \pi_1(X, x_0)$, then $j \circ h(\sigma) = \hat{\sigma} = c_{x_0} * \sigma * \overline{c_{x_0}} = \sigma$. Now we wish to show that $h \circ j = id$. Let $c = \sum_i n_i \sigma_i$ be a 1-cycle in X. Then $h \circ j$ takes c to $\sum_i n_i h(\hat{\sigma_i}) = \sum_i n_i(\varphi_{p_i} + \sigma_i - \varphi_{q_i})$, where we say σ_i goes from p_i to q_i. Then we can split this sum into two pieces, $\sum_i n_i(\varphi_{p_i} + \sigma_i - \varphi_{q_i}) = \sum_i n_i \sigma_i + \sum_i n_i(\varphi_{p_i} - \varphi_{q_i})$ and note that this first sum is in fact c. Thus, it suffices to show that $\sum_i n_i(\varphi_{p_i} - \varphi_{q_i}) = 0$.

Now c is a 1-cycle so $\partial c = 0$, but $\partial c = \sum_i n_i(p_i - q_i)$. Regroup the terms of this last sum into a linear combination of distinct 0-simplices with integer coefficients. This sum can only be 0 if each of the coefficients are 0. Now $\sum_i n_i(\varphi_{p_i} - \varphi_{q_i})$ can be written as a linear cominbation of distinct 1-simplices with the same coeffecients as before and thus, as a 0 combination. $\square$

Tensor Products

Let A, B be abelian groups. Define the abelian group

$$A \otimes B = \{a \otimes b | a \in A,\ b \in B\} / \sim$$

where $\sim$ is generated by the relations $(a + a') \otimes b = a \otimes b + a' \otimes b$ and $a \otimes (b + b') = a \otimes b + a \otimes b'$. The zero element of $A \otimes B$ is $0 \otimes b = a \otimes 0 = 0_{A \otimes B}$ since $0 \otimes b = (a + (-a)) \otimes b = a \otimes b - a \otimes b = 0_{A \otimes B}$. Similarly, the inverse of an element $a \otimes b$ is $-(a \otimes b) = (-a) \otimes b = a \otimes (-b)$.

Properties

(1) $A \otimes B \cong B \otimes A$ via the isomorphism $a \otimes b \mapsto b \otimes a$.
Proof: The map $\varphi : A \times B \to B \otimes A$ defined by $(a, b) \mapsto b \otimes a$ is clearly bilinear and therefore induces a homomorphism $\overline{\varphi} : A \otimes B \to B \otimes A$ with $a \otimes b \mapsto b \otimes a$. Similarly, there is the reverse map $\psi : B \times A \to A \otimes B$ defined by $(b, a) \mapsto a \otimes b$

which induces a homomorphism $\overline{\psi} : B \otimes A \to A \otimes B$ with $b \otimes a \mapsto a \otimes b$. Clearly, $\overline{\varphi} \circ \overline{\psi} = id_{B \otimes A}$ and $\overline{\psi} \circ \overline{\varphi} = id_{A \otimes B}$ and $A \otimes B \cong B \otimes A$.

(2) $(\bigoplus_i A_i) \otimes B \cong \bigoplus_i (A_i \otimes B)$ via the isomorphism $(a_i)_i \otimes b \mapsto (a_i \otimes b)_i$.

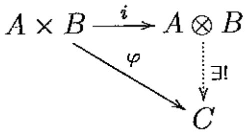
(3) $A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$ via the isomorphism $a \otimes (b \otimes c) \mapsto (a \otimes b) \otimes c$.

(4) $\mathbb{Z} \otimes A \cong A$ via the isomorphism $n \otimes a \mapsto na$.
Proof: The map $\varphi : \mathbb{Z} \times A \to A$ defined by $n \otimes a \mapsto na$ is a bilinear map and therefore induces a homomorphism $\overline{\varphi} : \mathbb{Z} \otimes A \to A$ with $n \otimes a \mapsto na$. Now suppose $\overline{\varphi}(n \otimes a) = 0$. Then $na = 0$ and $n \otimes a = 1 \otimes (na) = 1 \otimes 0 = 0_{\mathbb{Z} \otimes A}$ and $\overline{\varphi}$ is injective. Moreover, if $a \in A$, then $\overline{\varphi}(1 \otimes a) = a$ and $\overline{\varphi}$ is surjective as well.

(5) $\mathbb{Z}/n\mathbb{Z} \otimes A \cong A/nA$ via the isomorphism $l \otimes a \mapsto la$.
Proof: The map $\varphi : \mathbb{Z}/n\mathbb{Z} \times A \to A/nA$ defined by $(l, a) \mapsto la$ is a bilinear map and therefore induces a homomorphism $\overline{\varphi} : \mathbb{Z}/n\mathbb{Z} \otimes A \to A/nA$ with $l \otimes a \mapsto la$. Now suppose $\overline{\varphi}(l \otimes a) = la = 0$. Then $la = \sum_{i=1}^{k} na_i$ and $l \otimes a = 1 \otimes (la) = 1 \otimes (\sum_{i=1}^{k} na_i) = \sum_{i=1}^{k} (n \otimes a_i) = 0_{\mathbb{Z}/n\mathbb{Z} \otimes A}$ and $\overline{\varphi}$ is injective. Now let $a \in A/nA$. Then $\overline{\varphi}(1 \otimes a) = a$ and $\overline{\varphi}$ is surjective as well.

(6) A bilinear map $\varphi : A \times B \to C$ induces a homomorphism $\overline{\varphi} : A \otimes B \to C$ by $a \otimes b \mapsto \varphi(a, b)$.

We can in fact define the tensor product through the universal property that if $\varphi : A \times B \to C$ is any bilinear map, then there exists a unique map say $\overline{\varphi} : A \otimes B \to C$ such that $\varphi = \overline{\varphi} \circ i$ where $i : A \times B \to A \otimes B$ is the natural map $(a, b) \mapsto a \otimes b$.



More generally, if R is a ring and A and B are R-modules:

(1) if R is commutative, define the R-module $A \otimes_R B := A \otimes B / \sim$, where $\sim$ is the relation generated by $ra \otimes b = a \otimes rb = r(a \otimes b)$.

(2) if R is not commutative, we need A a right R-module and B a left R-module and the relation is $ar \otimes b = a \otimes rb$. In this case $A \otimes_R B$ is only an abelian group.

In both cases, $A \otimes_R B$ is not necessarily isomorphic to $A \otimes B$.

Example. Let $R = \mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$. Now $R \otimes_R R \cong R$ which is a 2-dimensional $\mathbb{Q}$-vector space. However, $R \otimes R$ as a $\mathbb{Z}$-module is a 4-dimesnional $\mathbb{Q}$-vector space.

Exercise.

- Find $\mathbb{Z}/n\mathbb{Z} \otimes \mathbb{Z}/m\mathbb{Z}$
- Show $\mathbb{Q} \otimes \mathbb{Z}/n\mathbb{Z} = 0$.

Lemma If G is an abelian group, then $-\otimes G$ is right exact, that is if $A \xrightarrow{i} B \xrightarrow{j} C \to 0$ is exact, then $A \otimes G \xrightarrow{i \otimes 1_G} B \otimes G \xrightarrow{j \otimes 1_G} C \otimes G \to 0$ is exact.

Proof: Let $c \otimes g \in C \otimes G$. Since j is onto, there exists, $b \in B$ such that $j(b) = c$. Then $j \otimes 1_G(b \otimes g) = c \otimes g$ and $j \otimes 1_G$ is onto.

Since $j \circ i = 0$, we have $(j \otimes 1_G) \circ (i \otimes 1_G) = (j \circ i) \otimes 1_G = 0$ and thus, $im(i \otimes 1_G) \subseteq ker(j \otimes 1_G)$.

It remains to show that $ker(j \otimes 1_G) \subseteq im(i \otimes 1_G)$. It is enough to show that

$$\psi : \frac{B \otimes G}{im(i \otimes 1_G)} \xrightarrow{\cong} C \otimes G,$$

where ψ is the map induced by $j \otimes 1_G$. Construct an inverse $\varphi : C \times G \to B \otimes G/im(i \otimes 1_G)$ by $(c, g) \mapsto b \otimes g$ where $j(b) = c$. We must show that φ is a well-defined bilinear map and that the induced map $\overline{\varphi}$ satisfies $\overline{\varphi} \circ \psi = id$ and $\psi \circ \overline{\varphi} = id$.

If $j(b) = j(b') = c$ then $b - b' \in ker\ j = im\ i$ and $b - b' = i(a)$ for some $a \in A$. Thus, $b \otimes g - b' \otimes g = (b - b') \otimes g = i(a) \otimes g \in im(i \otimes 1_G)$ and φ is well defined.

Now $\varphi((c + c', g)) = d \otimes g$ where $j(d) = c + c'$. Since j is surjective, choose $b, b' \in B$ such that $j(b) = c$ and $j(b') = c'$. Then $d - (b + b') \in ker\ j = im\ i$ and so there exists $a \in A$ such that $i(a) = d - (b + b')$. Thus, $\varphi((c + c', g)) = d \otimes g = (b + b') \otimes g = b \otimes g + b' \otimes g = \varphi(c, g) + \varphi(c', g)$ and φ is linear in the first component. For the second component, $\varphi(c, g + g') = b \otimes (g + g') = b \otimes g + b \otimes g' = \varphi(c, g) + \varphi(c, g')$. Thus, φ is bilinear.

Now by property (6) of the tensor product, the bilinear map φ induces a homomorphism $\overline{\varphi} : C \otimes G \to \frac{B \otimes G}{im(i \otimes 1_G)}$ where $c \otimes g \mapsto \varphi(c, g)$. For $c \otimes g \in C \otimes G$, $\psi \circ \overline{\varphi}(c \otimes g) = \psi(b \otimes g) = j(b) \otimes g = c \otimes g$ and $\psi \circ \overline{\varphi} = id_{C \otimes G}$. Similarly, for $b \otimes g \in \frac{B \otimes G}{im(i \otimes 1_G)}$, $\overline{\varphi} \circ \psi(b \otimes g) = \overline{\varphi}(j(b) \otimes g) = \varphi(j(b), g) = b \otimes g$ and $\overline{\varphi} \circ \psi = id$. □

Universal Coeffecient Theorem for Homology

Goal: Compute $H_*(X; G)$ in terms of $H_*(X; \mathbb{Z})$ and G.

Problem: Given a chain complex $C_\bullet : \cdots \to C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots \to C_0 \to 0$ of free abelian groups and G an abelian group, compute $H_*(C_\bullet; G) = H_*(C_\bullet \otimes G)$ in terms of $H_*(C_\bullet; \mathbb{Z})$ and G.

Answer

Theorem (Universal Coefficient Theorem)
There are natural short exact sequences:

$$0 \to H_n(C_\bullet) \otimes G \to H_n(C_\bullet; G) \to \text{Tor}(H_{n-1}(C_\bullet), G) \to 0 \text{ for all } n.$$

By natural we mean: if $C_\bullet \to C'_\bullet$ is a chain map, then there is an induced map of short exact sequences with commuting squares. Moreover, these short exact sequences split, but not naturally.

In particular, if $C_\bullet = C_*(X, A)$ is a singular chain complex, then there are natural short exact sequences

$$0 \to H_n(X, A) \otimes G \to H_n(X, A; G) \to \text{Tor}(H_{n-1}(X, A), G) \to 0.$$

Naturality is with respect to maps of pairs $(X, A) \xrightarrow{f} (Y, B)$. There are splittings, but not naturally.

Say $A = B = \emptyset$, $f : X \to Y$ and $H_n(X; G) = H_n(X) \otimes G \oplus \text{Tor}(H_{n-1}(X), G)$, $H_n(Y; G) = H_n(Y) \otimes G \oplus \text{Tor}(H_{n-1}(Y), G)$. If these splittings were natural, and f induces the trivial map $f_* = 0$ on $H_*(-; \mathbb{Z})$ then f induces the trivial map on $H_*(-; G)$ which is a contradiction as we saw a counterexample in the previous week.

Definition A free resolution of an abelian group H is an exact sequence:

$$\cdots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0,$$

with each F_n free abelian.

Given an abelian group G, from a free resolution $F_\bullet$ of H, we can obtain a modified chain complex:

$$F_\bullet \otimes G : \cdots \to F_2 \otimes G \to F_1 \otimes G \to F_0 \otimes G \to 0.$$

We define $\mathrm{Tor}_n(H, G) := H_n(F_\bullet \otimes G)$. Note here that we have removed the final term of the complex to account for the fact that $- \otimes G$ is right exact.

Lemma For any two free resolutions $F_\bullet$ and $F'_\bullet$ of H there are canonical isomorphisms $H_n(F_\bullet \otimes G) \cong H_n(F'_\bullet \otimes G)$ for all n. Thus, $\mathrm{Tor}_n(H, G)$ is independent of the free resolution $F_\bullet$.

Given an abelian group H, take F_0 to be the free abelian group on a set of generators of H to get $F_0 \twoheadrightarrow H \to 0$. Let $F_1 := ker\ f_0$, and note that F_1 is a free group, as it is a subgroup of a free abelian group F_0. Let $F_1 \hookrightarrow F_0$ be the inclusion map. Then

$$0 \to F_1 \hookrightarrow F_0 \twoheadrightarrow H \to 0,$$

is a free resolution of H.

Thus, $\mathrm{Tor}_n(H, G) = 0$ if $n > 1$. Moreover, $\mathrm{Tor}_0(H, G) \cong H \otimes G$. We can thus adopt the notation that $\mathrm{Tor}_1(H, G) := \mathrm{Tor}(H, G)$.

Properties of Tor

(1) $\mathrm{Tor}(A, B) = \mathrm{Tor}(B, A)$.

(2) $\mathrm{Tor}(\bigoplus_i A_i, B) = \bigoplus_i \mathrm{Tor}(A_i, B)$.

(3) $\mathrm{Tor}(A, B) = 0$ if A or B is free of torsion-free.

(4) $\mathrm{Tor}(\mathrm{T}(A), B) = \mathrm{Tor}(A, B)$, where $\mathrm{T}(A)$ is the torsion subgroup of A.

(5) $\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z}, A) \cong ker\ (A \xrightarrow{n} A)$.

Corollary $\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) = \frac{\mathbb{Z}}{(n,m)\mathbb{Z}} = \mathbb{Z}/n\mathbb{Z} \otimes \mathbb{Z}/m\mathbb{Z}$, where (n, m) is the greatest common divisor of n and m.
Corollary If A and B are finitely generated abelian groups, then

$$\mathrm{Tor}(A, B) = \mathrm{T}(A) \otimes \mathrm{T}(B)$$

where $\mathrm{T}(A)$ and $\mathrm{T}(B)$ are the torsion subgroups of A and B respectively.

(6) For a short exact sequence: $0 \to B \to C \to D \to 0$ of abelian groups, there is a natural exact sequence:

$$\begin{aligned} 0 \to &\mathrm{Tor}(A, B) \to \mathrm{Tor}(A, C) \to \mathrm{Tor}(A, D) \to \\ &A \otimes B \to A \otimes C \to A \otimes D \to 0. \end{aligned}$$

Proof of (5): $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$ is a free resolution of $\mathbb{Z}/n\mathbb{Z}$. Now $- \otimes A$ gives $\mathbb{Z} \otimes A \xrightarrow{n \otimes 1_A} \mathbb{Z} \otimes A \to 0$ which by property (4) of the tensor product is $A \xrightarrow{n} A \to 0$. Thus, $\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z}, A) = ker\ (A \xrightarrow{n} A)$.