Algebraic Topology I
TKMD Summer School
METU Math. Dept. June 20–24, 2016

Introduction:

1) Definition of topology: Open, closed, closure, interior.

2) Examples
- Some finite topologies
- $\mathbb{R}_{std}$, $\mathbb{R}^n_{std}$
- Real line with double origin

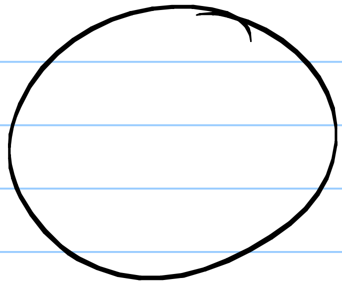
3) Equivalence of topologies:
- Continuity
- Homeomorphism
- Homeomorphic spaces will be regarded the same

4) Basis, subbasis of a topology. Example $\mathbb{R}^n_{std}$.

5) Strong / Weak Topology

6) New topologies from old ones.

- Subspace topology



$S^1 \subseteq \mathbb{R}^2$

- Topological Embedding

<u>Example:</u> $X = [0, \infty)$

$$\mathcal{B} = \{(a,b) \mid 0 < a < b\} \cup \{[0,a) \cup (b,\infty) \mid a, b > 0\}.$$

Let $\mathcal{T}$ be the topology on X generated by $\mathcal{B}$. Then

$f : [0,\infty) \longrightarrow S^1$, $f(t) = e^{2\pi i t/1+t}$

is a homeomorphism.

<u>Example</u> $X = \mathbb{R}$

$\mathcal{B}' = \{(a,b) \mid ab > 0\}$
$\cup \{(a,b) \cup (c_1, \infty) \cup (-\infty, c_2) \mid$
$a < 0 < b\}$

If τ' is the topology on $\mathbb{R}$ generated by $\mathcal{B}'$ then (X, τ') is homeomorphic to the figure 8 in the plane.

- <u>Product Topology</u>
  - $(X_\alpha, \tau_\alpha)_{\alpha \in \Lambda}$ a family of spaces

$$\prod_\alpha X_\alpha \doteq \{f : \Lambda \to \dot{\bigcup} X_\alpha \mid f(\alpha) \in X_\alpha\}.$$

(Axiom of Choice!)

- Box Topology
- Comparison of the two topologies.

<u>Proposition</u>: A function

$f: X \to \prod_\alpha X_\alpha$ is continuous

if and only if each

$f_\alpha : X \to X_\alpha$ is continuous.

<u>Remark</u> This is <u>not</u> true

if we put the box topology on $\prod_\alpha X_\alpha$.

<u>Example</u> $f: \mathbb{R} \to \mathbb{R}^{\mathbb{N}}$

$t \mapsto (t, t, t, \dots)$

- <u>Quotient Topology</u>

- Equivalence relations on a set X

$$\Big\updownarrow \text{ 1-1 correspondence}$$

Surjective functions
$\pi: X \longrightarrow Y$.

- If $\sim$ is an equivalence relation on X and $\pi: X \to X/\sim$, $f(x) = [x]$ then π is called a quotient map.

Let $f: X \longrightarrow Z$ be another map.

<u>Question:</u> Is there a map $g: X/\sim \longrightarrow Z$ s.t. the diagram below is commutative

$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[dr, "f"] & \\ X/\sim \arrow[r, "g"'] & Z \end{tikzcd}$$

?

<u>Answer</u>: There is such g if and only if f is constant on the fibers of π: If $[x]=[y]$ for some $x, y \in X$ then $f(x)=f(y)$. Moreover, in this case g is unique!

- Now let τ be a topology on X. Then there is a unique topology on $X/\sim$ so that whenever $f: X \to Z$ is a continuous map which is constant on the fibers of $\pi: X \to X/\sim$ then g is continuous. This topology is given by:

$U \subseteq X/\sim$ is open if and only if $\pi^{-1}(U)$ is open.

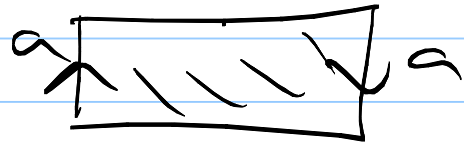
- This is the weakest topology on $X/\sim$ making $\pi: X \to X/\sim$ continuous.

Example: $\mathbb{RP}^n = \mathbb{R}^{n+1} \setminus \{0\} / x \sim \lambda x$

$x \in \mathbb{R}^{n+1} \setminus \{0\}$, $\lambda \in \mathbb{R} \setminus \{0\}$.

$\mathbb{CP}^n = \mathbb{C}^{n+1} \setminus \{0\} / x \sim \lambda x$

$x \in \mathbb{C}^{n+1} \setminus \{0\}$, $\lambda \in \mathbb{C} \setminus \{0\}$.

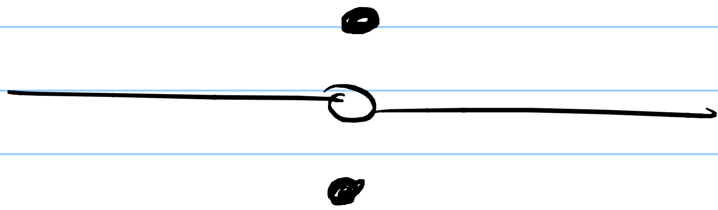
Example 1) MB: 

2) T^2: 

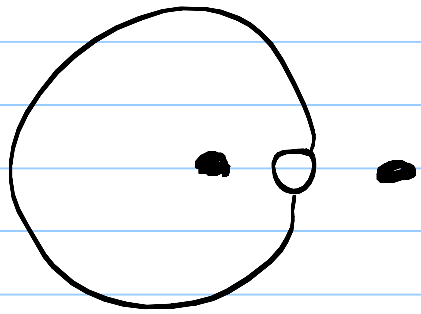
3) $\mathbb{RP}^2$: 

7) Separation Axioms:
T_0, T_1, T_2, T_3 and T_4.

Examples and counter examples.

Example 

and

X: 

$f: S^1 \longrightarrow X$ degree two map
(local homeomorphism!)

8) <u>Compactness:</u>

- General definition
- Sequential compactness for metric spaces.
- $\mathbb{R}$, $\mathbb{R}^n$

<u>Theorem</u> (Heine–Borel)
A subspace X of $\mathbb{R}^n$ is compact if and only if A is closed and bounded.

- Product of compact spaces are compact: Tychonoff Theorem

<u>Theorem</u> (Homeomorphism)
A continuous bijection $f: X \to Y$ is a homeomorphism provided that the spaces are Hausdorff and X is compact.

9) Connectedness:

- Definition
- Arc connectedness

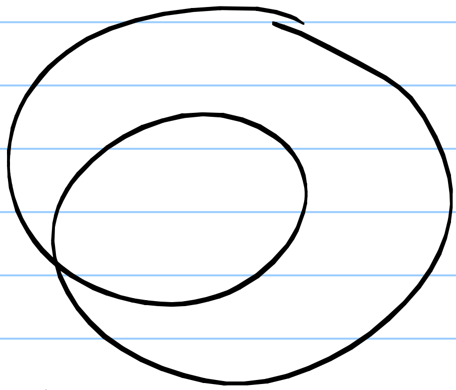
Example Topologist Sine curve.

Example: An embedding of $\mathbb{RP}^2$ into $\mathbb{R}^5/\mathbb{R}^4$.

(10) Why algebraic Topology?

• Comparing topological objects is just to difficult.

__Example__ How to distinguish

 and  ?

They are different embeddings of $S^1 \vee S^1$ = figure eight.

• Algebraic objects are easier to compare, like numbers.
The first has rotation number 2 and the second has rotation number zero.

In general topological classification is too difficult, so we use a coarser classification; Homotopy equivalence:

11) • Definition of Homotopy of maps / Relative homotopy.

- Homotopy equivalence
- Retraction $r: X \Rightarrow A$
- Deformation Retraction of a space X onto a subspace A

$f: X \times I \longrightarrow X$ cont. map s.t.

1) $f(x,0) = x, \forall x \in X$
2) $f(a,t) = a, \forall a \in A, \forall t \in [0,1]$.
3) $f(x,1) \in A, \forall x \in X$.

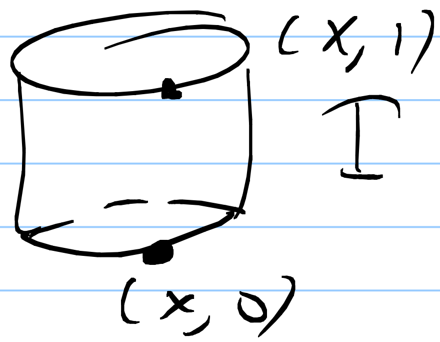
- Contractibility: $1_X \simeq *$.

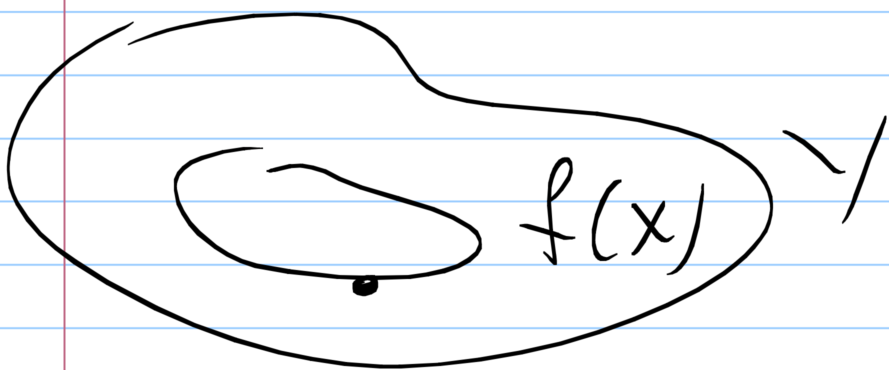
Remark: X deformation-retracts onto a subspace A then X and A are homotopy equivalent.

12) Mapping Cylinder

$f: X \to Y$ any map

$$M_f = X \times I \sqcup Y \Big/ {(x,0) \sim f(x)}$$





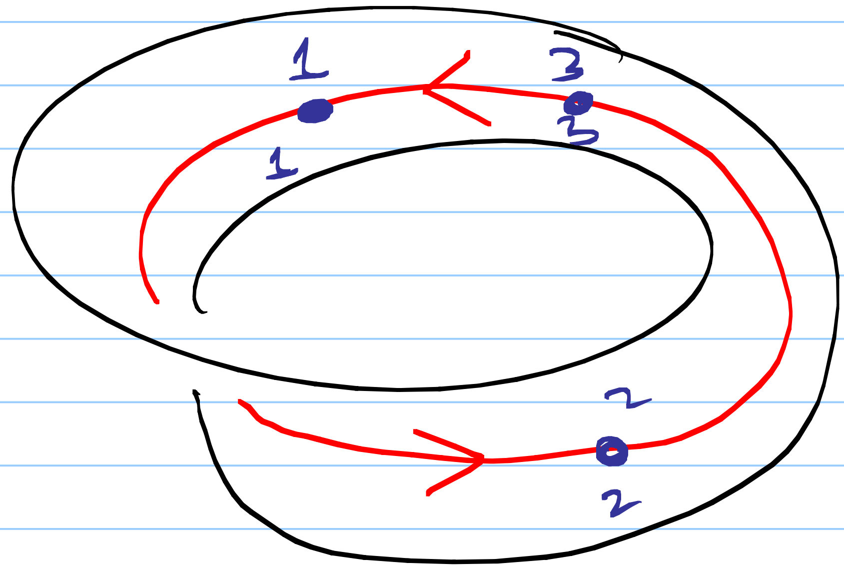
Remark

M_f deformation retracts onto Y.

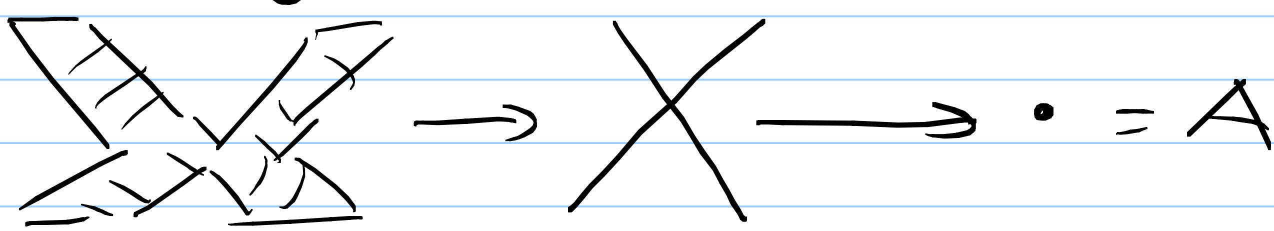
Example $f: S^1 \to S^1$
$z \mapsto z^2$

$M_f = MB$.

Proof: Cut through a MB along the center circle.

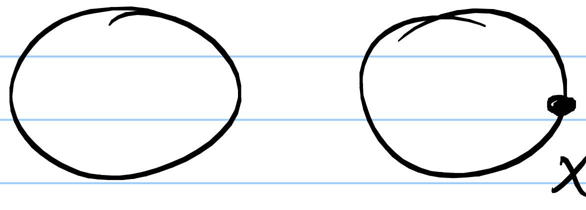


Remarks 1) Not all deformation retractions are obtained from mapping cylinders.



Certain pairs of points follow paths that merge before they reach final destination. This does not happen in a mapping cylinder.

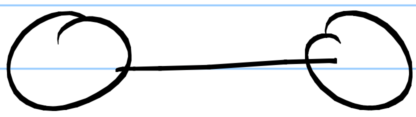
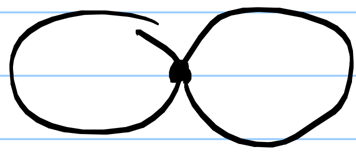
2) Not all retractions come from deformation retractions.

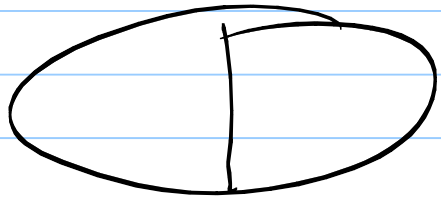
X:  $\longrightarrow \{x_0\}$

is a retraction. However, if $f: X \times I \to X$ is a deformation retraction onto $A = \{x_0\}$ then X must be path connected!

3) Homotopy equivalence is

an equivalence relation but deformation retraction is not:

,  and

 are all deformation retractions of the same space $\mathbb{R}^2 \setminus \{(\pm 1, 0)\}$ but they are <u>not</u> deformation retractions of each other. (Exercise!)

<u>Corollary 0.21</u> Two spaces X and Y are homotopy equivalent if and only if there is a third space Z which deformation retracts onto X and Y.

13) Cell complexes: We build a space inductively as follows:

1) Start with a discrete set of points X^0, whose elements are called 0-cells.

2) Form the n-skeleton X^n from X^{n-1} by attaching n-cells e^n_α via maps

$$\varphi_\alpha : S^{n-1} = \partial D^n \longrightarrow X^{n-1}.$$

Hence, $X^n = X^{n-1} \coprod_\alpha D^n_\alpha \Big/ {x \sim \varphi_\alpha(x) \atop x \in \partial D^n}$

where e^n_α is $\mathrm{Int}(D^n_\alpha)$.

3) One can stop at some stage n and let $X = X^n$ or continue indefinitely, setting

$X = \cup X^n$. In this case, a subset $A \subset X$ is called open (closed) if and only if $X^n \cap A$ is open (resp. closed in X^n).

Such a space is called a CW-complex.

C: closure finite

W: weak topology

If $X = X^n$ then we say that X has dimension n.

Examples S^n, T^2, KB, S^∞, 1-dim'l CW complex = Graph

For any n-cell e^n_α the map $\varphi_\alpha : D^n_\alpha \longrightarrow X^n \longrightarrow X$ is called the characteristic map. The map $X^n \longrightarrow X$ is continuous since it has the weak topology. The map $D^n_\alpha \longrightarrow X^n$ is continuous since it is the union of an inclusion via a quotient map.

__Remark__ Any CW-complex is the disjoint union of its cells.

__Definition__ 1) If each characteristic map $\varphi_\alpha : D^n_\alpha \longrightarrow X$ is an embedding then the complex D called a regular complex.

2) Simplicial complex when each D^n_α is Δ^n and gluing maps are linear homeomorphisms.

14) Euler characteristic of a finite cell complex.

15) <u>Operation on Spaces</u>

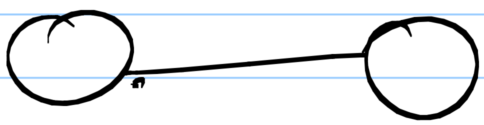
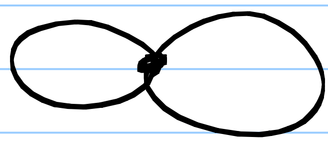
- Product of cell complexes
- Quotients $A \subseteq X$ subcomplex then X/A has a complex structure.
- Suspension
- Cone on X
- Join of X and Y : $X * Y$

- Wedge $X \vee Y$
- Smash product: $X \times Y / X \vee Y$

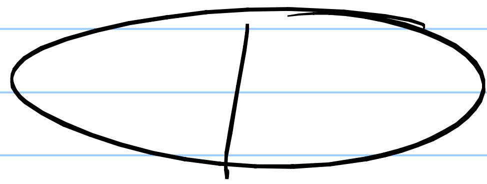
$$S^1 \times S^1 / S^1 \vee S^1 \simeq S^2$$

$$S^n \times S^m / S^n \vee S^m \simeq S^{n+m}$$

16) <u>Theorem</u> If (X, A) is a CW pair where A is contractible that $X \longrightarrow X/A$ is a homotopy equivalence.

<u>Example</u>  $\underset{h.e.}{\simeq}$ 

is h.e.



<u>Example</u> Any connected finite graph is homotopy

equivalent to $\bigvee_n S^1$ for some n.

Theorem If (X_1, A) is a CW pair and two attaching maps $f, g : A \to X_0$ are homotopic then $X_0 \amalg_f X_1$ homotopy equivalent to $X_0 \amalg_g X_1$.

Exercises

1) $S^\infty = \lim_n S^n$, $i_n : S^n \to S^{n+1}$, $x \mapsto (0, x)$

$$= \{ (x_1, x_2, \dots, x_n, \dots) \mid x_i \in \mathbb{R}$$

$x_i = 0$ for all but finitely many i and $\sum_i x_i^2 = 1 \}$

Show that S^∞ is contractible.

Solution: Let $T: S^\infty \to S^\infty$

$T((x_1, x_2, \dots, x_n, \dots)) = (0, x_1, x_2, \dots, x_n, \dots)$

Let $F_t(x) = \dfrac{(1-t)x + tT(x)}{\|(1-t)x + tT(x)\|}$, $t \in [0,1]$.

$F_0 = Id_{S^\infty}$ and $F_1(x) = T(x)$.

$F: S^\infty \times [0,1] \to S^\infty$, $F(x,t) = F_t(x)$.

Also, let $S_0^\infty = \{(x_i) \in S^n \mid x_1 = 0\}$

and $G: S_0^\infty \times [0,1] \to S^\infty$ by

$G(x,t) = \dfrac{t(1,0,0,\dots) + (1-t)x}{\|t(1,0,0,\dots) + (1-t)x\|}$.

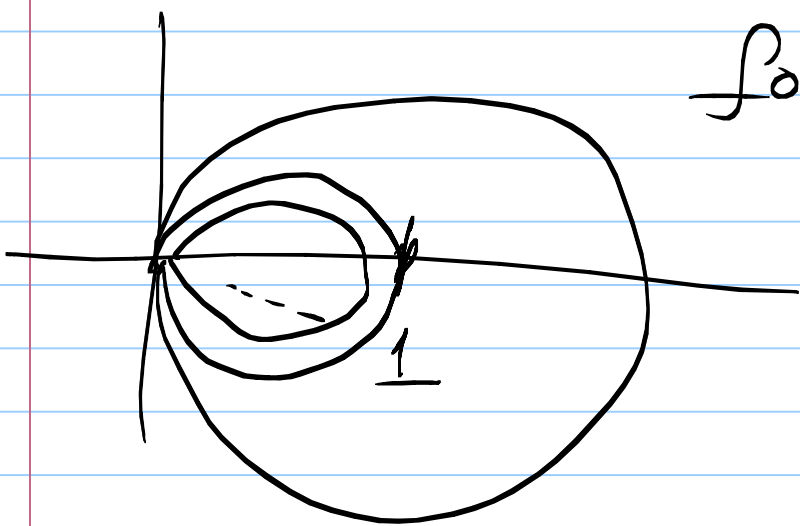
$G(x,0) = x$ and $G(x,1) = (1,0,0,\dots)$.

Finally, let $H: S^\infty \times [0,1] \to S^\infty$

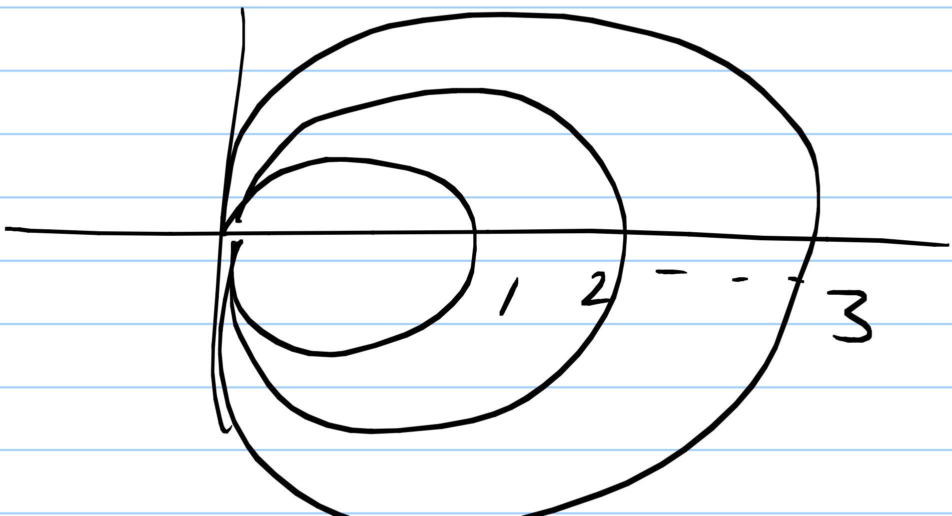
$$H(x,t) = \begin{cases} F(x,2t), & 0 \le t \le 1/2 \\ G(x, 2t-1), & 1/2 \le t \le 1 \end{cases}$$

2) $X_1 = \bigvee_{i=1}^{\infty} S^1$

$X_2 = \{(x,y) \in \mathbb{R}^2 \mid (x-\frac{1}{n})^2 + y^2 = \frac{1}{n^2}$ for some $n \in \mathbb{Z}^+\}$

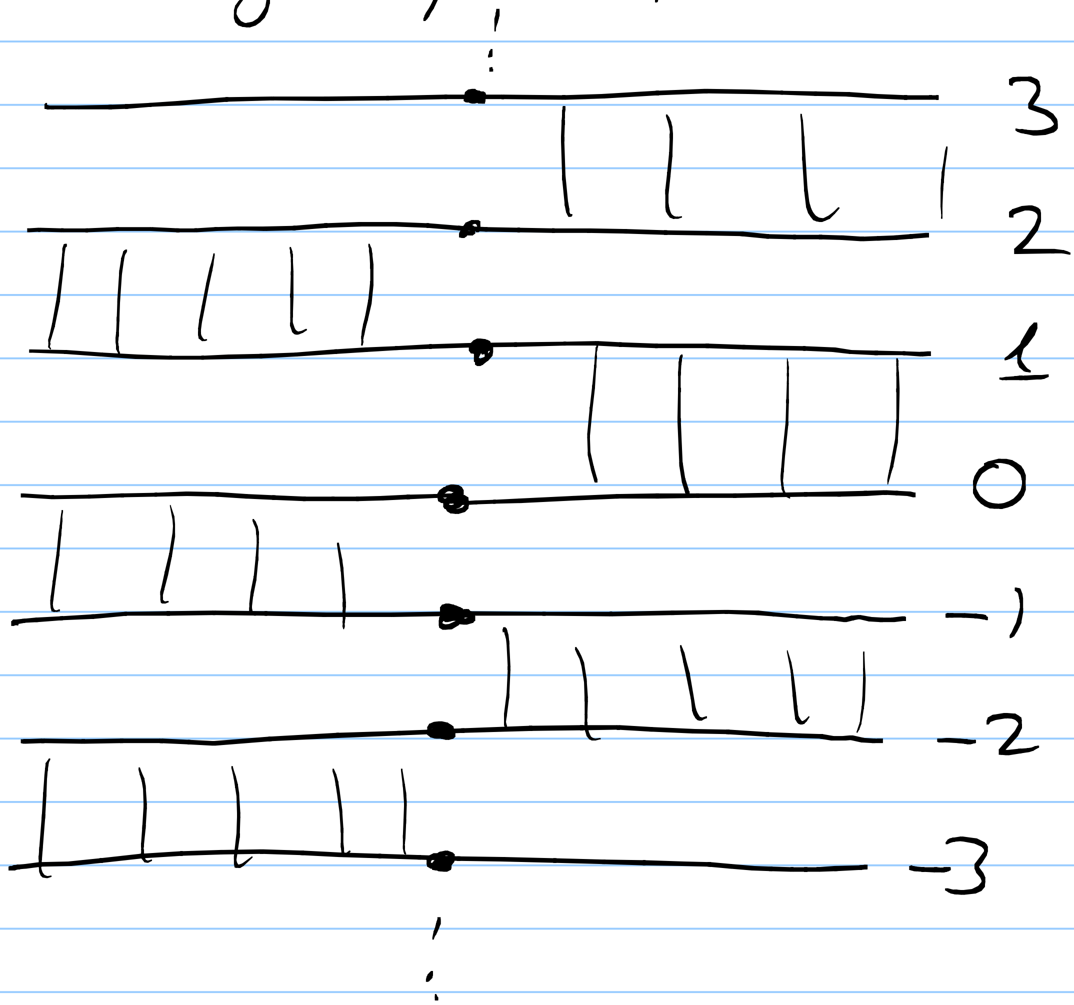


$X_3 = \{(x,y) \in \mathbb{R}^2 \mid (x-n)^2 + y^2 = n^2,$ for some $n \in \mathbb{Z}^+\}$.



Is X_i homeomorphic to X_j?

3) Let $X = \mathbb{R} \times \mathbb{Z}/\sim$ be the following space:

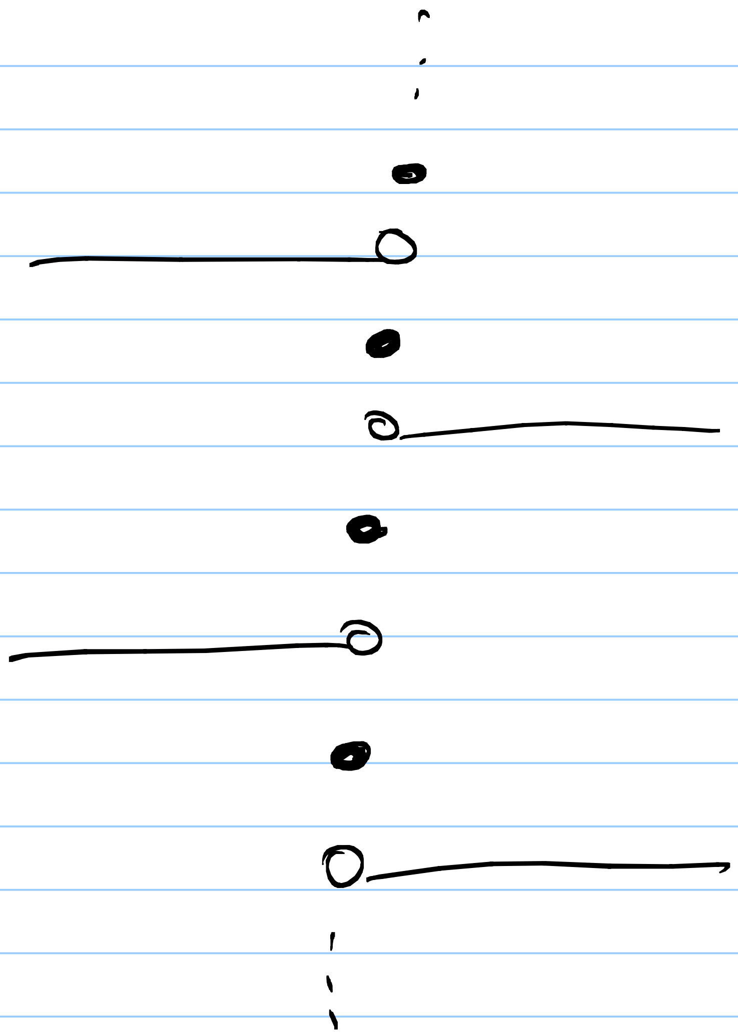


$(x, n) \sim (x, n+1)$ for $x > 0$ and n even

$(x, n) \sim (x, n+1)$ for $x < 0$ and n odd.

Hint: Note that each $\mathbb{R} \times \{n\}$ is open in $\mathbb{R} \times \mathbb{Z}$ and thus its image is open in X.

X:



Show that if K is a compact space and $f : K \to X$ is a continuous map then f is homotopic to a constant map.

The Fundamental Group

1) It is a functor from the category of topological spaces with based points to the category of groups:

$(X, x_0) \longmapsto \pi_1(X, x_0)$.

$x_0 \in X$

Definition: Given a based space (X, x_0) let $\mathcal{L}$ be the set of based loops at x_0:

$\mathcal{L} = \{\gamma : [0,1] \to X \text{ cont}, \gamma(0) = x_0, \gamma(1) = x_0\}$

Define the homotopy relation on $\mathcal{L}$: $\alpha, \beta \in \mathcal{L}$, $\alpha \sim \beta$ if and only if there is an homotopy $F : [0,1] \times [0,1] \to X$

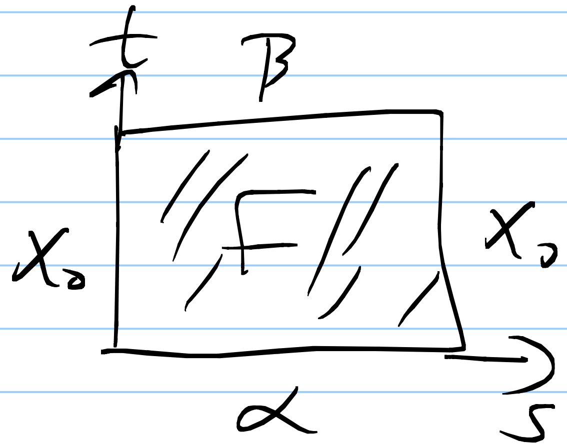
such that for all $0 \le s, t \le 1$,

i) $F(s,0) = \alpha(s)$

ii) $F(s,1) = \beta(s)$

iii) $F(0,t) = x_0$

iv) $F(1,t) = x_0$



Define $\pi_1(X, x_0)$ as the set of equivalence classes $\Omega/\sim$.

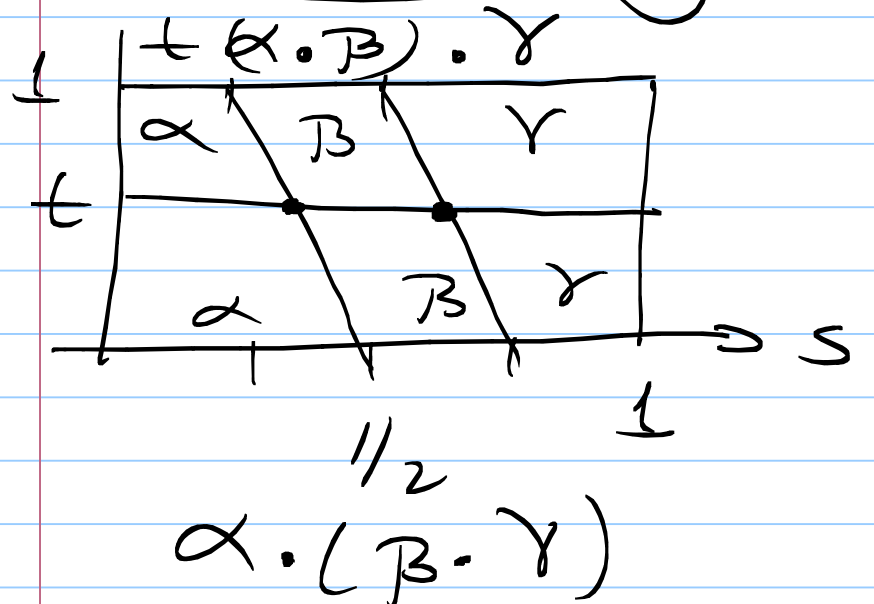
<u>Group structure on π_1</u>:

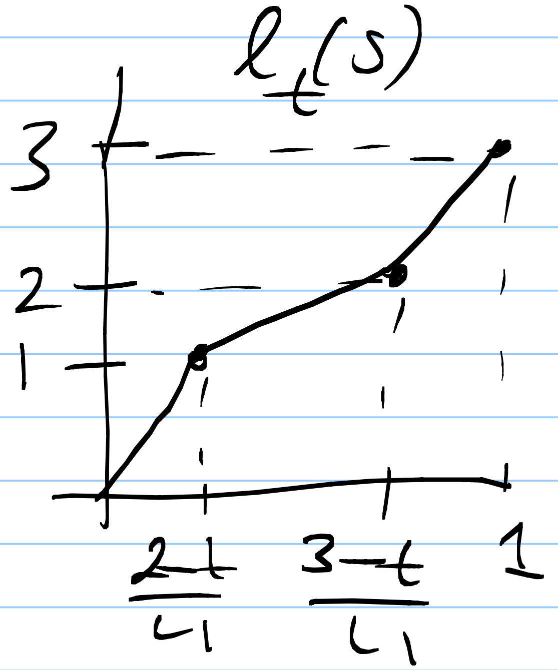
For $\alpha, \beta \in \Omega$ let $\alpha \cdot \beta$ denote the product loop

$$(\alpha \cdot \beta)(s) = \begin{cases} \alpha(2s), & 0 \le s \le 1/2 \\ \beta(2s-1), & 1/2 \le s \le 1. \end{cases}$$

This descends to a multiplication on $\pi_1(X, x_0)$ which makes π_1 a group with the identity element $[e]$, where $e:[0,1]\to X$, $e(s)=x_0$, the constant loop.

Associativity:





Let
$$l_t(s)=\begin{cases}\dfrac{4s}{2-t}, & 0\le s\le \dfrac{2-t}{4}\\[2mm] 4s+t-1, & \dfrac{2-t}{4}\le s\le \dfrac{3-t}{4}\\[2mm] \dfrac{4s+3t-1}{1+t}, & \dfrac{3-t}{4}\le s\le 1\end{cases}$$

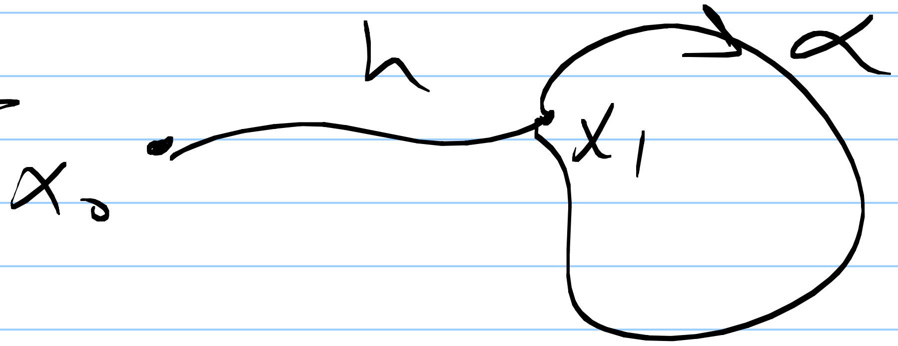
$$H(s,t)=\begin{cases}\alpha(\ell_t(s)) & 0\le s\le \frac{2-t}{4}\\ \beta(\ell_t(s)-1) & \frac{2-t}{4}\le s\le \frac{3-t}{4}\\ \gamma(\ell_t(s)-2) & \frac{3-t}{4}\le s\le 1.\end{cases}$$

Example For a convex subset X of $\mathbb{R}^n_{std}$ and any $x_0\in X$ $\pi_1(X,x_0)=(e)$.

Proposition: Let X be a space $x_0, x_1 \in X$ points and $h:[0,1]\to X$ is a path joining x_0 to x_1: $h(0)=x_0$, $h(1)=x_1$. Then the map $\beta_h:\pi_1(X,x_1)\to\pi_1(X,x_0)$ defined by

$\beta_h([\alpha])=[h\cdot\alpha\cdot\bar{h}]$ is an is an isomorphism.

Proof



Definition: A path connected space X is called simply connected if $\pi_1(X, x_0) = \{e\}$ for some (and thus all) $x_0 \in X$.

Proposition: A space X is simply connected if and only if for any two point $x_0 \in X$ and $x_1 \in X$ there is a unique homotopy class of paths joining x_0 to x_1, where homotopies fix the end points at all times.

<u>Remark:</u> If X is simply connected and $x_0 \in X$ then there is a bijection

$$\{\alpha : [0,1] \to X \mid \alpha(0) = 1\} \big/ \sim$$

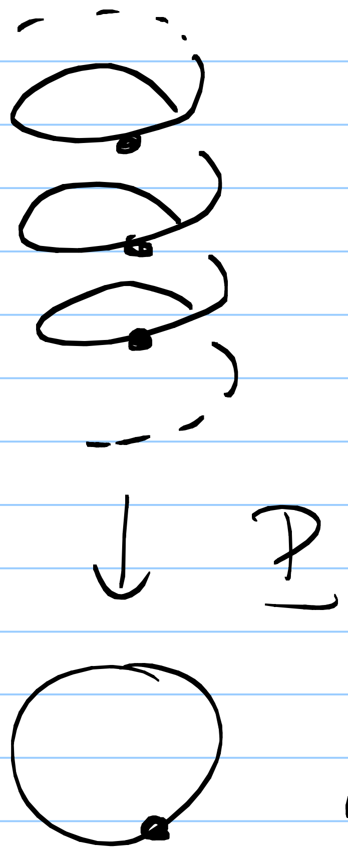
$$\begin{array}{ccc} \tilde{\alpha} & & \\ \downarrow & & \updownarrow \\ \alpha(1) & \in & X \end{array}$$

when $\alpha \sim \beta$ if and only if they are homotopic through homotopies fixing the end points.

2) The fundamental Group of the circle

Theorem: The map $\Phi: \mathbb{Z} \to \pi_1(S^1)$ sending an integer n to the homotopy class of the loop
$\omega_n(s) = (\cos 2\pi ns, \sin 2\pi ns)$ based at $(1,0)$ is an isomorphism.

Proof: i) Consider the map
$p: \mathbb{R} \to S^1$, $p(s) = (\cos 2\pi ns, \sin 2\pi ns)$



Let $\tilde{\omega}_n(s) = ns$ satisfy
$\omega_n(s) = p(\tilde{\omega}_n(s))$.
$\tilde{\omega}_n$ is a lift of ω_n.

Note that $\Phi(n)$ can be defined as the homotopy class of the loop $p \circ \tilde{f}$ for any path $\tilde{f}$ in $\mathbb{R}$ from 0 to n, because any such $\tilde{f}$ is homotopic to $\tilde{\omega}_n$, keeping end points fixed: $t \longmapsto (1-t)\tilde{f} + t\tilde{\omega}_n$ and thus $p \circ \tilde{f}$ is homotopic to ω_n:

$$[p \circ \tilde{f}] = [\omega_n].$$

ii) <u>Claim:</u> ϕ is a homomorphism.

<u>Proof</u> Let $\tau_m : \mathbb{R} \to \mathbb{R}$ be the translation $\tau_m(x) = x + m$. The $\tilde{\omega}_m \cdot (\tau_m(\tilde{\omega}_n))$ is a path in $\mathbb{R}$ from 0 to $m+n$, so $\Phi(m+n)$ is the homotopy class of a loop in S^1, which is the image of this path under p. This image is just $\omega_m \cdot \omega_n$, so that

$$\phi(m+n) = \Phi(m) \cdot \phi(n).$$

To show Φ is an isomorphism we shall use two facts:

a) For each path $f: I \to S^1$ starting at a point $x_0 \in S^1$ and each $\tilde{x}_0 \in p^{-1}(x_0)$ there is a unique lift $\tilde{f}: I \to \mathbb{R}$ starting at $\tilde{x}_0$.

b) For each homotopy $f_t: I \to S^1$ of paths starting at x_0 and $\tilde{x}_0 \in p^{-1}(x_0)$ there is a unique lifted homotopy $\tilde{f}_t: I \to \mathbb{R}$ of path starting at $\tilde{x}_0$.

(ii) (a) and (b) prove the theorem.

<u>Φ is surjective</u>: Let $f: I \to S^1$ be a loop at the base point $(1,0)$ representing a given element of $\pi_1(S^1)$. By (a) there is a lift $\tilde{f}$

starting at 0. This path $\tilde{f}$ ends at some integer since $p \circ \tilde{f}(1) = f(1) = (1,0)$ and $p^{-1}(1,0) = \mathbb{Z} \subseteq \mathbb{R}$. By the extended definition of Φ we have $\Phi(n) = [p\tilde{f}] = [f]$. Hence, Φ is surjective.

<u>Φ is injective</u>: Suppose $\Phi(m) = \Phi(n)$, which implies $\omega_m \simeq \omega_n$. Let f_t be a homotopy from $\omega_m = f_0$ to $\omega_n = f_1$. By (b) this homotopy lifts to a homotopy $\tilde{f}_t$ of paths starting at 0. The uniqueness part of (a) implies that $\tilde{f}_0 = \tilde{\omega}_m$ and $\tilde{f}_1 = \tilde{\omega}_n$. Since $\tilde{f}_t$ is a homotopy of paths the endpoints $\tilde{f}_t(1)$

is independent of t. For $t=0$ the endpoint is m and for $t=1$ it is n, so $m=n$.

Instead of (a) and (b) we'll prove

<u>(c)</u>: Given a map $f: Y \times I \to S^1$ and a map $\tilde{F}: Y \times \{0\} \to \mathbb{R}$ lifting $F\big|_{Y \times \{0\}}$, then there is a unique map $\tilde{F}: Y \times I \to \mathbb{R}$ lifting F and restricting to the given $\tilde{F}$ on $Y \times \{0\}$.

(c) $\Rightarrow$ (a): $Y = \{pt\}$.

(c) $\Rightarrow$ (b) $Y = I$

In the proof of (c) we make use the following property

of the map $p : \mathbb{R} \to S^1$:
There is an open cover $\{U_\alpha\}$ of S^1 so that for each α, $p^{-1}(U_\alpha)$ is a disjoint union of open subsets each of which is homeomorphically mapped onto U_α via p.

<u>Defn:</u> Let $p : X \to Y$ be an onto map. If Y has an open cover $\{U_\alpha\}$ so that for each α, $p^{-1}(U_\alpha)$ is a disjoint union of open subsets each of which is mapped homeomorphically onto U_α then p is called a covering map.

3) Proof of c) First construct a lift $\tilde{F} : N \times I \to \mathbb{R}$ for N

some neighborhood in Y of a point $y_0 \in Y$. Since F is continuous every point $(y_0, t) \in Y \times I$ has a product neighborhood $N_t \times (a_t, b_t)$ s.t. $F(N_t \times (a_t, b_t)) \subset U_\alpha$ for some α. By compactness of $\{y_0\} \times I$ finitely many such products cover $\{y_0\} \times I$, say $N_{\alpha_i} \times (a_i, b_i)$ $i = 1, \ldots, k$. Let $N = \bigcap_{i=1}^{k} N_{\alpha_i}$

and $0 = t_0 < t_1 < t_2 < \cdots < t_m = 1$.

$F(N \times [t_i, t_{i+1}]) \subseteq U_i \doteq U_{\alpha_i}$.

Assume that $\tilde{F}$ has been constructed on $N \times [0, t_i]$. Since $F(N \times [t_i, t_{i+1}]) \subseteq U_i$ there

is some $\widetilde{U}_i \subseteq \mathbb{R}$ projecting homeomorphically onto U_i by p and containing the point $\widetilde{F}(y_0, t_i)$. If necessary replace N by a smaller open neighborhood so that $\widetilde{F}(N \times \{t_i\}) \subseteq \widetilde{U}_i$.
(Replace $N \times \{t_i\}$ with $N \times \{t_i\} \cap (\widetilde{F}|_{N \times \{t_i\}})^{-1}(\widetilde{U}_i)$.)

Now define $\widetilde{F}$ on $N \times [t_i, t_{i+1}]$ to be the composition of F with the homeomorphism $p^{-1} : U_i \longrightarrow \widetilde{U}_i$. Repeating this finitely many times we get the required lift $\widetilde{F} : N \times I \to \mathbb{R}$.

Now let's prove the uniqueness part of (c) in the case $Y=\{p\}$. We'll drop Y from the notation. So suppose $\tilde{F}$ and $\tilde{F}'$ are two lifts of $F: I \longrightarrow S^1$ so that $\tilde{F}(0)=\tilde{F}'(0)$. As above choose a partition $0=t_0<t_1<\cdots<t_m=1$ of I so that $F([t_i,t_{i+1}]) \subseteq U_i$ for some i. Assume inductively that $\tilde{F}=\tilde{F}'$ on $[0,t_i]$. Since $[t_i,t_{i+1}]$ is connected $\tilde{F}([t_i,t_{i+1}])$ must lie in a single component of $p^{-1}(U_i)$, say $\tilde{U}_i$. Similarly, $\tilde{F}'([t_i,t_{i+1}])$ must lie in a single component say $\tilde{\tilde{U}}_i$.

But $\tilde{F}(t_i) = \tilde{F}'(t_i)$ and thus $\tilde{U}_i = \tilde{U}_i'$. Since p is injective on $\tilde{U}_i$ and $p \circ \tilde{F} = p \circ \tilde{F}'$ it follows that $\tilde{F} = \tilde{F}'$ on $[t_i, t_{i+1}]$. So by induction $\tilde{F} = \tilde{F}'$ on $[0, t_m] = [0, 1]$.

Finishing the proof: Since $\tilde{F}$'s constructed above on the sets $N \times I$ are unique when restricted to each segment $\{y\} \times I$, they must agree whenever two such $N \times I$'s overlap. Thus we get a well defined lift $\tilde{F}$ on all of $Y \times I$. This $\tilde{F}$ is continuous since

$\tilde{H}$ is continuous on each $N\times I$, and $\tilde{H}$ is unique since $\tilde{H}$ is unique on each segment $\{y\}\times I$.

4) <u>Applications</u>:

<u>Theorem</u>: Every nonconstant polynomial with coefficients in $\mathbb{C}$ has a root in $\mathbb{C}$.

<u>Proof</u> Let $P(z)=z^n+a_1z^{n-1}+\dots+a_n$ and assume that $P(z)$ has no roots in $\mathbb{C}$. Then for any real number $r\geq 0$ the formula

$$f_r(s)=\frac{P(re^{2\pi i s})/P(r)}{|P(re^{2\pi i s})/P(r)|}$$

defines a loop in the unit circle $S^1 \subseteq \mathbb{C}$, based at 1.
In fact, as r varies f_r is a homotopy of loops based at 1. f_0 is the trivial loop at 1.
Hence $[f_r] = [f_0] = 0 \in \pi_1(S^1)$ the trivial element.
Let $r = |a_1| + |a_2| + \cdots + |a_n| + 1$.
Now for $|z| = r$ we have

$$|z^n| = r^n = r\, r^{n-1}$$
$$> (|a_1| + \cdots + |a_n|)\,|z|^{n-1}$$
$$\geq |a_1 z^{n-1} + \cdots + a_{n-1} z + a_n|.$$

So the polynomial
$P_t(z) = z^n + t(a_1 z^{n-1} + \cdots + a_n)$ has no zero on the circle $|z| = r$ when $0 \leq t \leq 1$.

Now the formula

$$\frac{P_t(re^{2\pi i s})/P_t(r)}{| \quad \downarrow \quad |} \quad \text{for}$$

$t \in [0,1]$ defines a homotopy from w_n to f_r, where

$w_n(s) = e^{2\pi i n s}$ so

$0 = [f_r] = [w_n] = n$

$\Rightarrow p(z) = a_0$ is a constant.

$\blacksquare$

Theorem: Every continuous map $f: D^2 \to D^2$ has a fixed point

Proof (Brouwer proved this for D^n in 1910).

Easy!

Theorem (Borsuk–Ulam)
For every continuous map $f: S^2 \to \mathbb{R}^2$ there exists a pair of antipodal points x and $-x$ in S^2 with $f(x) = f(-x)$.

Proof Suppose $f(x) \neq f(-x)$ for all $x \in S^n$.
Define $g: S^2 \to S^1$ as $g(x) = \dfrac{f(x) - f(-x)}{\| f(x) - f(-x) \|}$.

Let $\eta: [0,1] \to S^1$, $\eta(s) = (\cos 2\pi s, \sin 2\pi s)$ and $h = g \circ \eta$. Since $g(-x) = -g(x)$ we have $h(s + \frac{1}{2}) = -h(s)$.

Let $\tilde{h}: [0,1] \to \mathbb{R}$ be a lift of h.

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ I \arrow[ur, "\tilde{h}"] \arrow[r, "h"'] & S^1 \end{tikzcd}$$

Since $h(s + \frac{1}{2}) = -h(s)$ we have
$$\tilde{h}(s + \tfrac{1}{2}) = \tilde{h}(s) + \frac{q}{2}$$

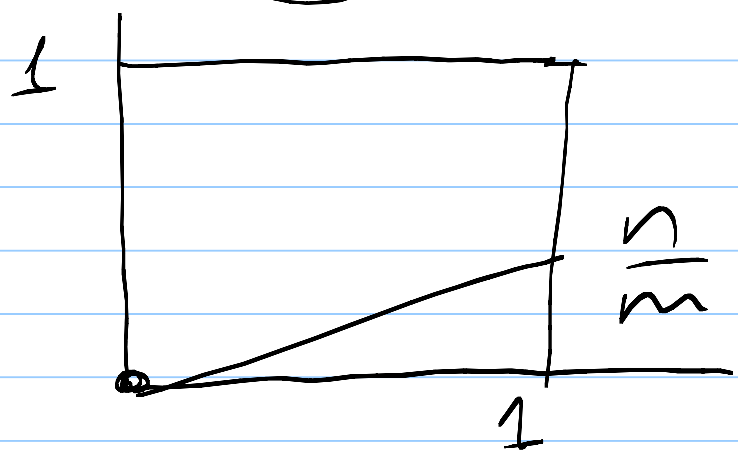
for some odd integer q. Note that since $q = 2(\tilde{h}(s+\frac{1}{2}) - \tilde{h}(s))$ and $\tilde{h}$ is continuous q is independent of s. In particular, $\tilde{h}(1) = \tilde{h}(\frac{1}{2}) + \frac{q}{2} = \tilde{h}(0) + q$

$\Rightarrow [h] = q \in \mathbb{Z} = \pi_1(S^1)$. Since q is an odd integer it is not zero and thus h is <u>not</u> null homotopic. However, $h = g \circ \eta$ and η is clearly null homotopic. Thus $h = g \circ \eta$ is null homotopic. This is a contradiction.

$\blacksquare$

Proposition: If X and Y are path connected then
$\pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y)$.

Corollary $\pi_1(S^1 \times S^1) = \mathbb{Z} \times \mathbb{Z}$.



$(m, n) \longrightarrow (e^{2\pi i s m}, e^{2\pi i s n})$

5) Induced Homomorphisms:

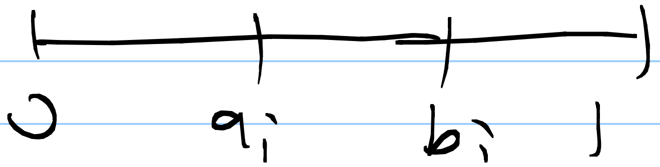
$\varphi : X \longrightarrow Y$, $x_0 \in X$, $y_0 = \varphi(x_0) \in Y$
$\Rightarrow \varphi_{\#} : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$.

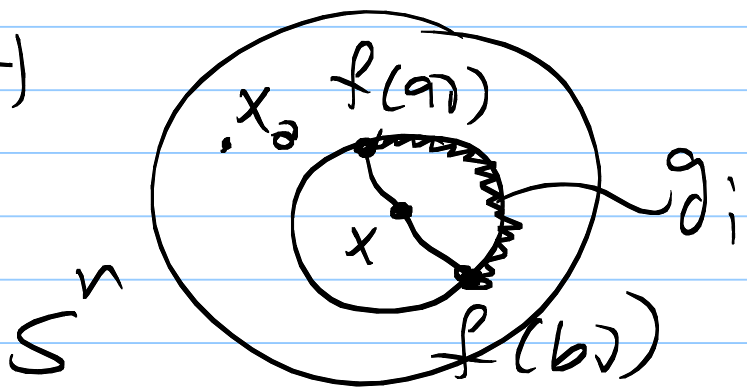
a) $\psi : (Y, y_0) \to (Z, z_0)$ then
$(\psi \circ \varphi)_{\#} = \psi_{\#} \circ \varphi_{\#}$

b) $1 = 1_X$, then $1_{\#} = 1_{\pi_1(X,x_0)}$.

6) <u>Theorem</u> $\pi_1(S^n, x_0) = (0)$, $n \geq 2$.

<u>proof</u> Let $f: [0,1] \to S^n$ be any loop based at x_0. Let $x \in S^n$, $x \neq x_0$ and choose a small open ball B^n around x. Then $f^{-1}(B^n)$ is a disjoint union of intervals in $(0,1)$. Let (a_i, b_i) be one of these intervals. Then $f_i \sim$
$f: [a_i, b_i] \to B$, $f(a_i), f(b_i) \in \partial B^n$.





One can then find a homotopy $f|_{[a_i, b_i]}$ to some $g_i : [a_i, b_i] \to S^n$

so that $g([a_i, b_i]) \subseteq \partial B^n$ and $f(a_i) = g(a_i)$, $f(b_i) = g(b_i)$.

Repeating this process for each i we homotope f to some g so that $g(I)$ misses $x \in S^n$. $\Rightarrow$ $g(I) \subseteq S^n \setminus \{x\} = \mathbb{R}^n$

$\Rightarrow g \simeq *$. ∎

<u>Examples</u> $x \in \mathbb{R}^n$

$\mathbb{R}^n \setminus \{\alpha\} \simeq \mathbb{R} \times S^{n-1}$

$$\Rightarrow \pi_1(\mathbb{R}^n \setminus \{\alpha\}) = \begin{cases} \mathbb{Z} & \text{if } n = 2 \\ 0 & \text{if } n > 2. \end{cases}$$

<u>Corollary</u> $\mathbb{R}^2$ is <u>not</u> homeomorphic to $\mathbb{R}^n$.

Proposition: If a space X retracts onto a subspace A then the induced homomorphism

$i_*: \pi_1(A) \to \pi_1(X)$ is injective and

$r_*: \pi_1(X) \to \pi_1(A)$ is surjective, where $i: A \to X$ is the inclusion and $r: X \to A$ is the retraction.

Corollary S^1 is not a retract of D^2.

Proposition: Let $\varphi: X \to Y$ be a homotopy equivalence. Then $\varphi_{\#}: \pi_1(X, x_0) \to \pi_1(Y, \varphi(x_0))$ is an isomorphism.

7) Van Kampen's Theorem:

Free products of groups: Let $G_1, \dots, G_k$ be groups. Then the free product of $G_1, \dots, G_k$, denoted by $G_1 * G_2 * \dots * G_k$ is the set of elements of the form $g_1 g_2 \dots g_n$, $g_i \in G_j$ with obvious group operation.

Example 1) $\mathbb{Z}_2 = \langle a \rangle$, $\mathbb{Z}_2 = \langle b \rangle$

$\mathbb{Z}_2 * \mathbb{Z}_2 = \{1, a, b, ab, ba, aba, bab, \dots\}$

$$1 \to \underset{\ker f}{\underset{\parallel}{N}} \to \mathbb{Z}_2 * \mathbb{Z}_2 \xrightarrow{f} \mathbb{Z}_2 \to 0$$

$a \longmapsto 1$

$b \longmapsto 1$

2) $\mathbb{Z}_2 * \mathbb{Z}_3 = \langle a, b \mid a^2 = 1, b^3 = 1 \rangle$

$\mathbb{Z}_2 * \mathbb{Z}_3 \longleftrightarrow PSL(2, \mathbb{Z})$ isomorphism

$a \leftrightarrow \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $b \leftrightarrow \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$.

Universal Property of Free products:

Let $f_i : G_i \to K$ $i = 1, \ldots, k$ be homomorphisms. Then there exists a unique homomorphism $F : G_1 * G_2 * \cdots * G_k \to K$ such that $F \circ i_j = f_j$ for all j, where

$$i_j : G_j \longrightarrow G_1 * \cdots * G_k$$
$$g \longmapsto g$$

Group presentation:

$\mathbb{Z} = \langle a \mid - \rangle$

$\mathbb{Z} * \mathbb{Z} = \langle a, b \mid - \rangle$

$\mathbb{Z}_n = \langle a \mid a^n \rangle$

$S_3 = \langle \sigma, \tau \mid \sigma^2, \tau^3, \sigma\tau\sigma\tau \rangle$

$F_n = \mathbb{Z} * \mathbb{Z} * \cdots * \mathbb{Z} = \langle a_1, \dots, a_n \mid - \rangle$

$\mathbb{Z} \times \mathbb{Z} = \langle a, b \mid aba^{-1}b^{-1} \rangle$

Theorem (Seifert, Van Kampen)
Let X be a topological space and U, V path connected open subsets of X so that $U \cap V$ is also path connected. Then the map $\Phi : \pi_1(U) * \pi_1(V) \to \pi_1(X)$, where $\Phi(g) = i_{U\#}(g)$ $\forall g \in \pi_1(U)$ and $\Phi(g) = i_{V\#}(g)$ $\forall g \in \pi_1(V)$, is

surjective. Moreover, the kernel $N \trianglelefteq B$ is generated by all elements of the form $J_{U\#}(\omega) J_{V\#}(\omega)$, where $\omega \in \pi_1(U \cap V)$:

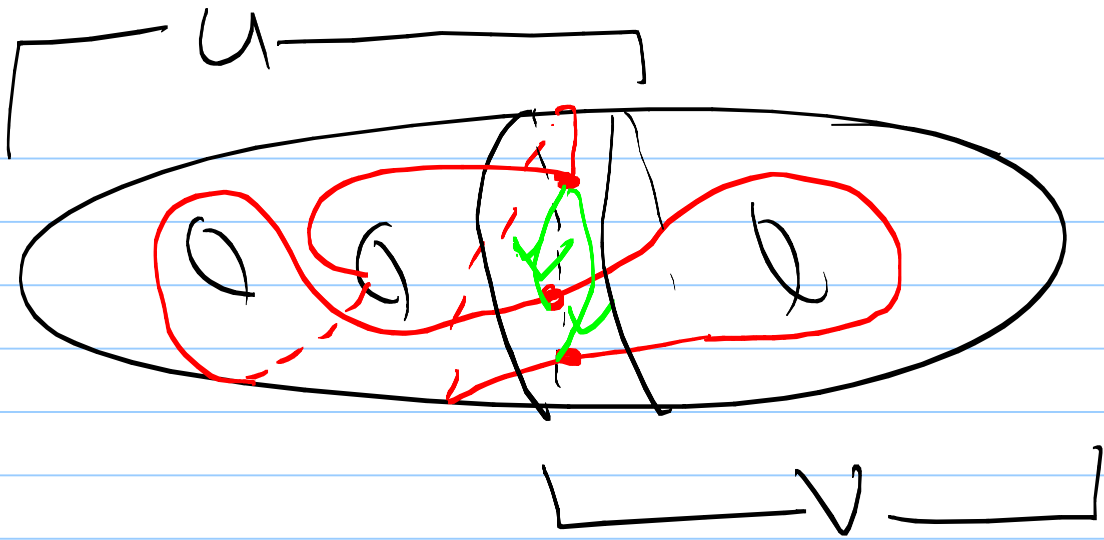
$$\pi_1(X) = \pi_1(U) \underset{\pi_1(U \cap V)}{*} \pi_1(V)$$

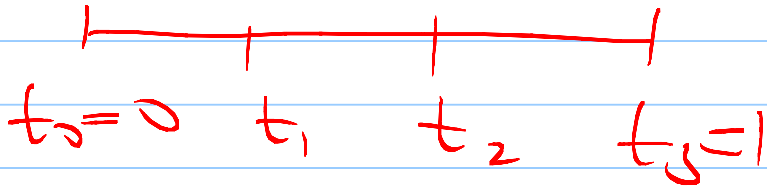
<u>OR</u>

$$\begin{tikzcd} & \pi_1(U) \arrow[dr, "I_{U\#}"] & \\ \pi_1(U \cap V) \arrow[ur, "J_{U\#}"] \arrow[dr, "J_{V\#}"'] & & \pi_1(U \cup V) \\ & \pi_1(V) \arrow[ur, "I_{V\#}"'] & \end{tikzcd}$$

<u>Idea of the proof of the first part:</u>

Given a map $f: [0,1] \to X$ with $f(0) = x_0 = f(1) \in U \cap V$ choose a partition of $[0,1]$ $0 = t_0 < t_1 < \dots < t_m = 1$ so that $f([t_i, t_{i+1}]) \subseteq U$ or $\subseteq V$.





$f \simeq f_1 \cdot f_2 \cdot f_3$

$f_1 \in \pi_1(U)$, $f_2 \in \pi_1(V)$, $f_3 \in \pi_1(U)$.

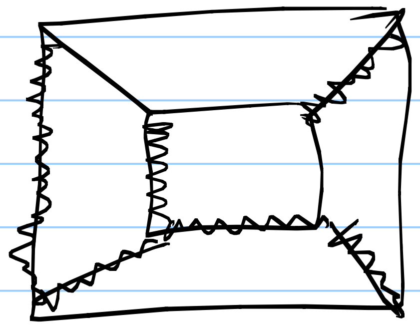
8) <u>Examples</u>: 1) $X = \bigvee_\alpha X_\alpha$

Suppose each X_α is path connected and $x_\alpha \in X_\alpha$ so that there is an open subset $x_\alpha \in U_\alpha \subseteq X_\alpha$ which deformation retracts onto $\{x_\alpha\}$.

Then $\pi_1(X) = \mathop{*}_\alpha \pi_1(X_\alpha)$

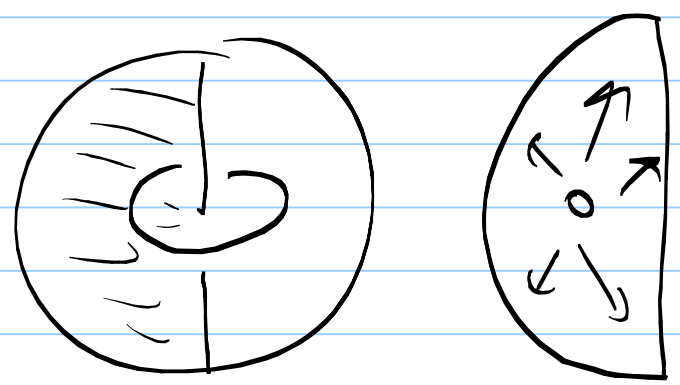
In particular, $\pi_1(\bigvee_n S^1) = \mathop{*}_n \mathbb{Z}$

2)



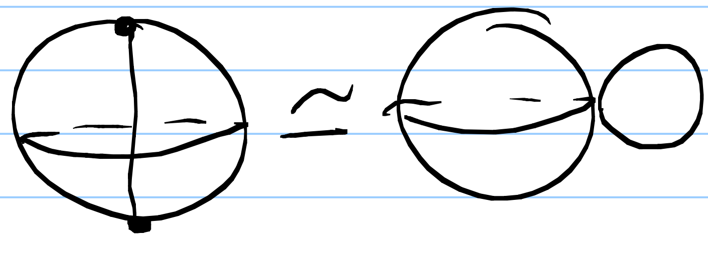
$\pi_1 \simeq \mathop{*}_5 \mathbb{Z}$

3) $\mathbb{R}^3 \setminus S^1 \simeq$



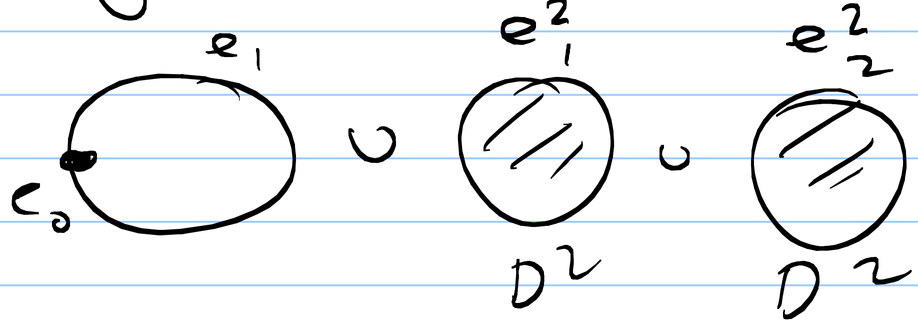
so $\mathbb{R}^3 \setminus S^1 = S^2 \vee S^1$

$\Rightarrow \pi_1 \simeq \mathbb{Z}$.



4) T^2, $\mathbb{RP}^2$, KB, Σ_g, N_h.

5) More general cell complexes.

6) $X =$ 

$\partial e_1^2 = S^1 \longrightarrow S^1 = e_0 \cup e_1$

$z \longmapsto z^2$

$\partial e_2^2 : S^1 \longrightarrow S^1$, $z \longmapsto z^2$.

$\pi_1(X) \simeq \mathbb{Z}_2$.

7) $\pi_1(\mathbb{RP}^2 \vee \mathbb{RP}^2) = \mathbb{Z}_2 * \mathbb{Z}_2$.

8) Construct a cell complex X such that $\pi_1(X) = \mathbb{Z}_2 * \mathbb{Z}_3$.

9) <u>Exercise</u>: Compute π_1 of spaces containing ∞'ly many circles!

COVERING SPACES

1) <u>Definition</u> A covering space of a space X is a space together with a map $p: \tilde{X} \to X$ so that there exists an open covering $\{U_\alpha\}$ of X, where each $p^{-1}(U_\alpha)$ is a disjoint union of open subsets of $\tilde{X}$ each of which is mapped homeomorphically onto U_α via p.

2) <u>Examples</u>

a)
$$\begin{array}{c} \mathbb{R} \\ \downarrow \; p(t) = e^{2\pi i t} \\ S^1 \end{array}$$

Regular
$G = \mathbb{Z}$

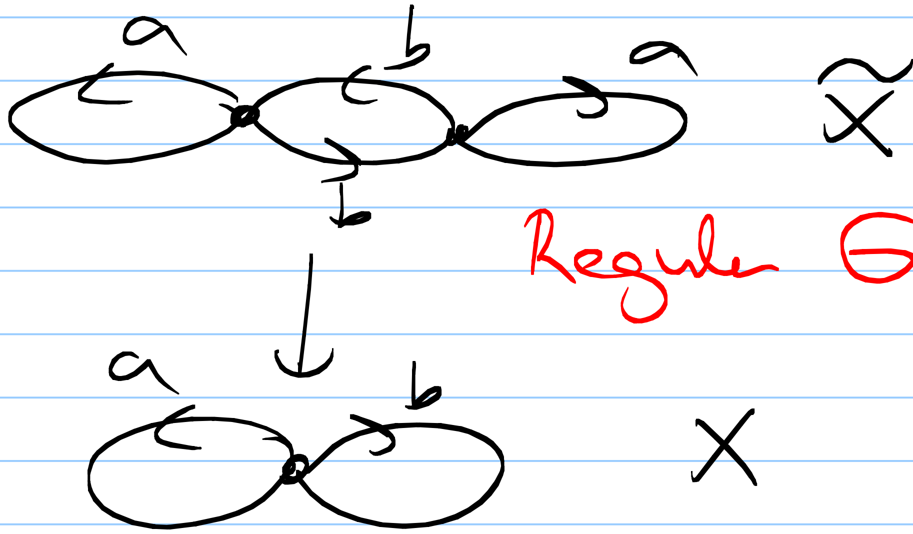
b) $p: S^1 \longrightarrow S^1$, $p(e^{2\pi i t}) = e^{2n\pi i t}$

$z \longmapsto z^n$ Regular $G = \mathbb{Z}_n$

c) $p: S^n \longrightarrow \mathbb{RP}^n = S^n / _{p \sim -p}$

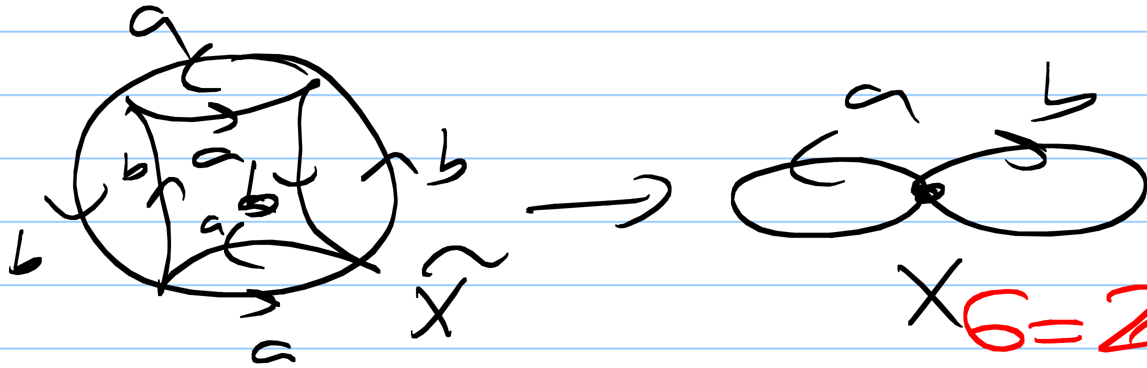
$p \longmapsto [p]$ Regular $G = \mathbb{Z}_2$

d)



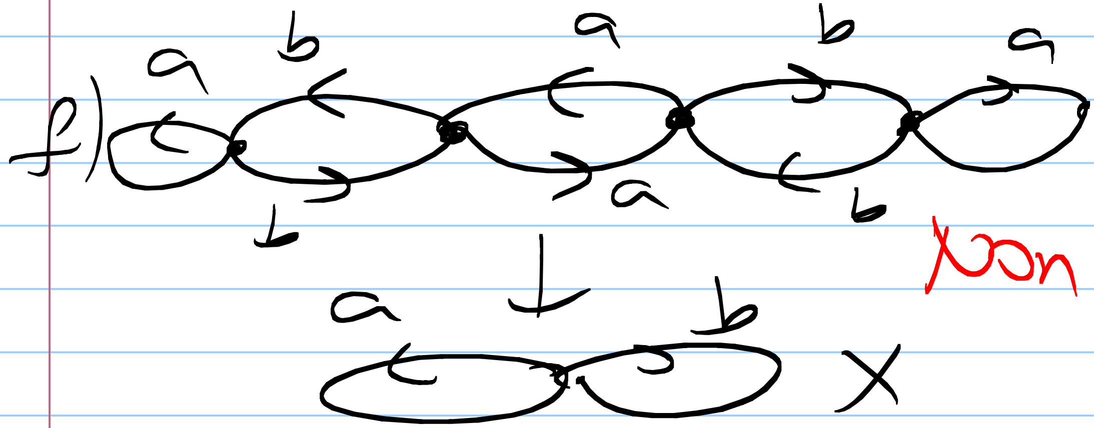
Regular $G = \mathbb{Z}_2$

e)

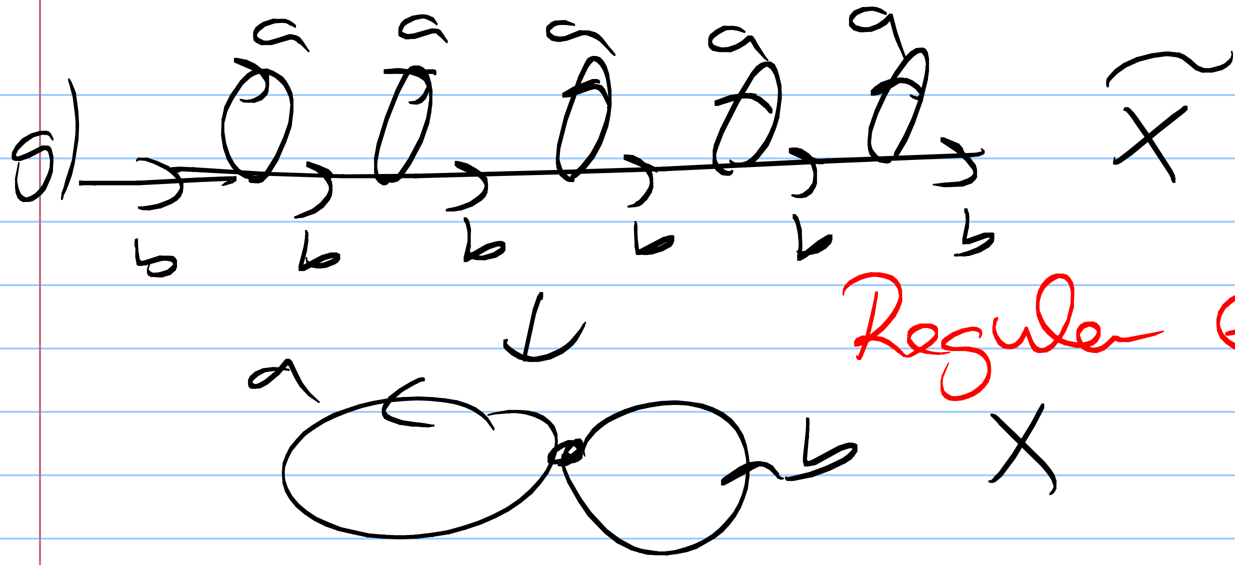


$G = \mathbb{Z}_2 \times \mathbb{Z}_2$

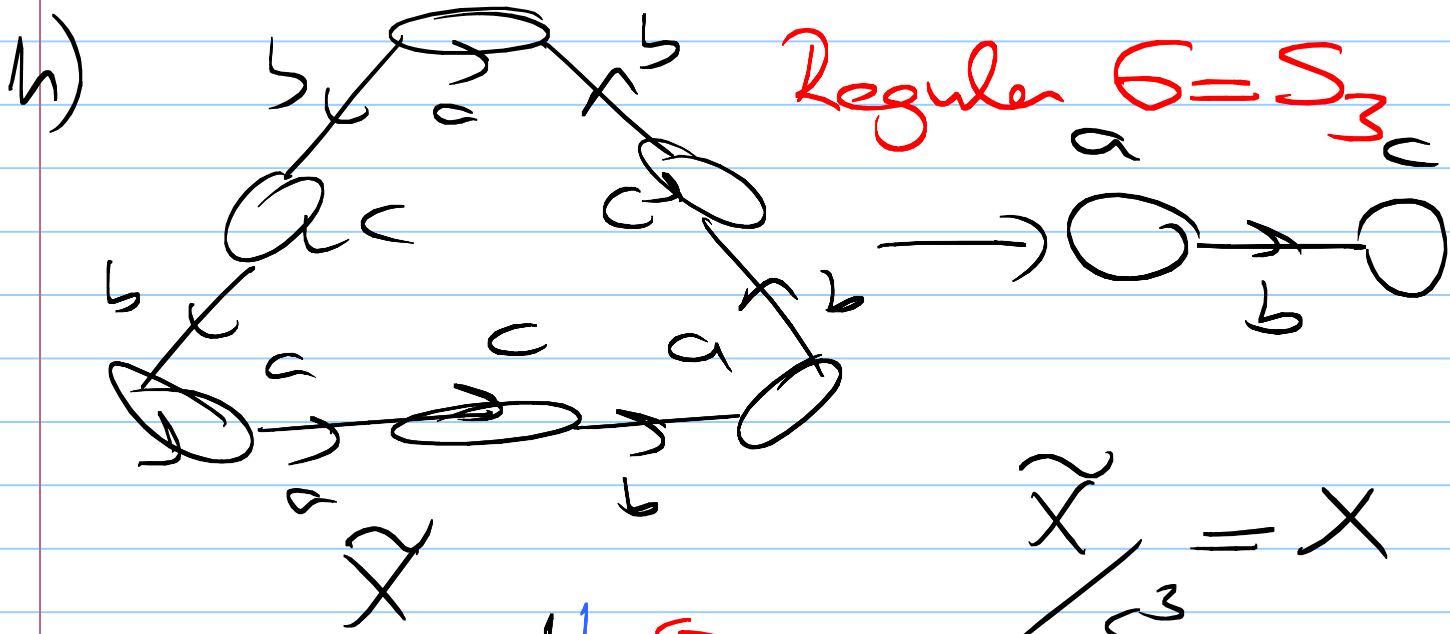
f)



Non regular

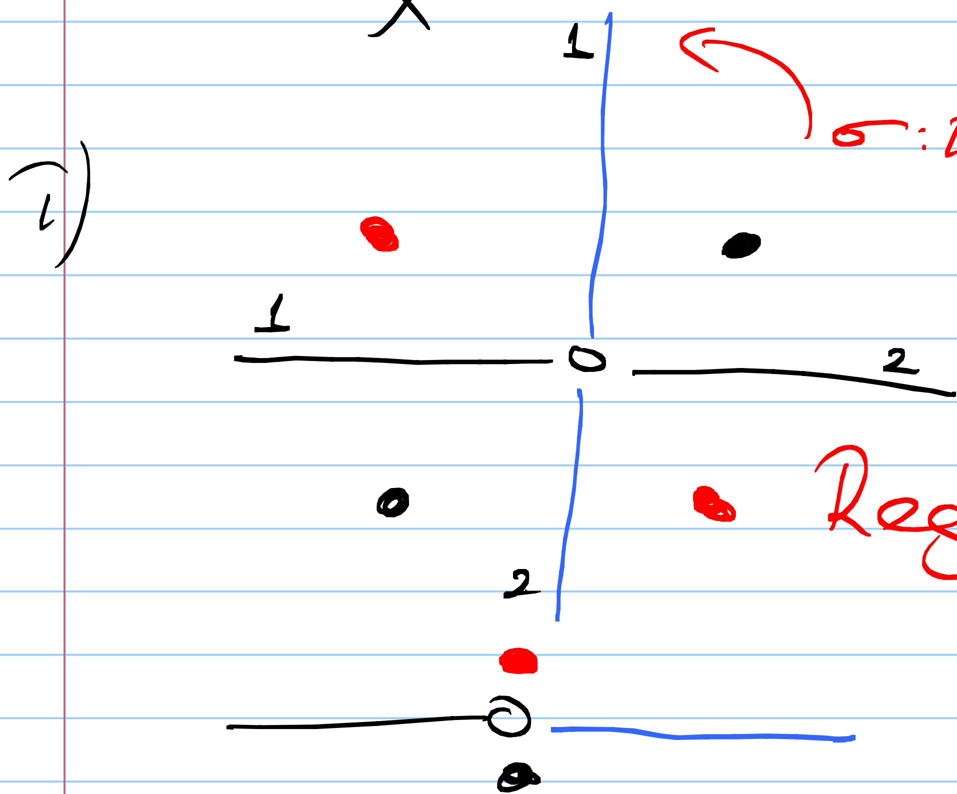


Regular $G = \mathbb{Z}$



Regular $G = S_3$

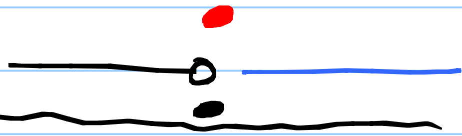
$\tilde{X} / S^3 = X$

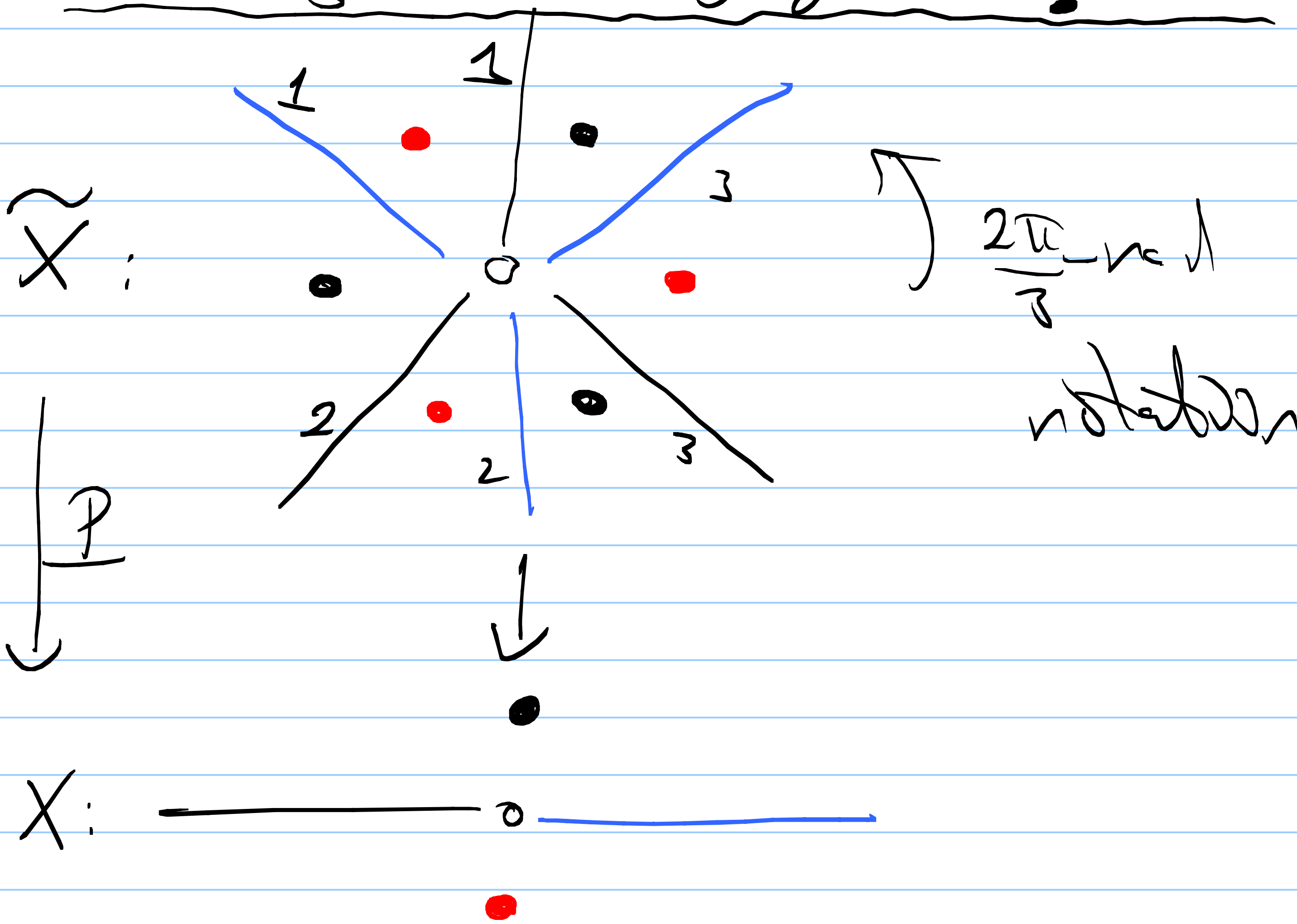


σ: π-rad rotation

Regular

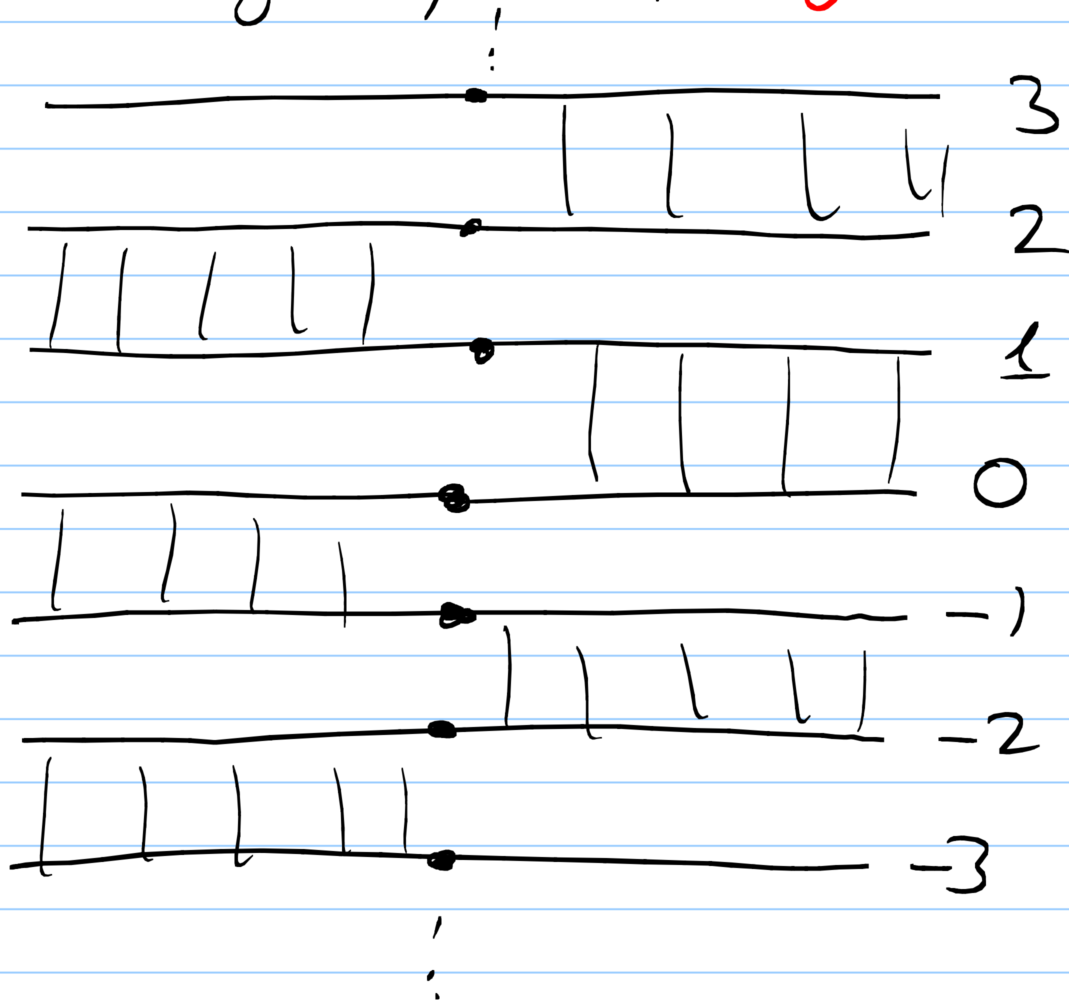
$G = \mathbb{Z}_2 = \langle \sigma \rangle$

A $\mathbb{Z}_3$-covering of 



Both spaces have $\pi_1 \simeq \mathbb{Z}$.

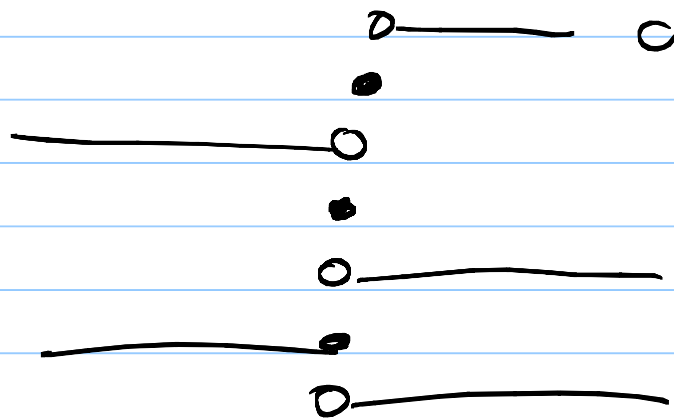
7) Let $\tilde{X} = \mathbb{R} \times \mathbb{Z} / \sim$ be the following space: Regular $G = \mathbb{Z}$



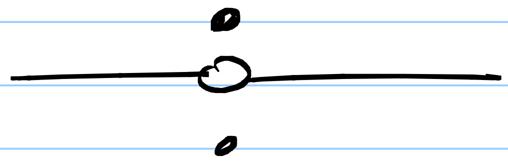
$(x, n) \sim (x, n+1)$ for $x > 0$ and n even

$(x, n) \sim (x, n+1)$ for $x < 0$ and n odd

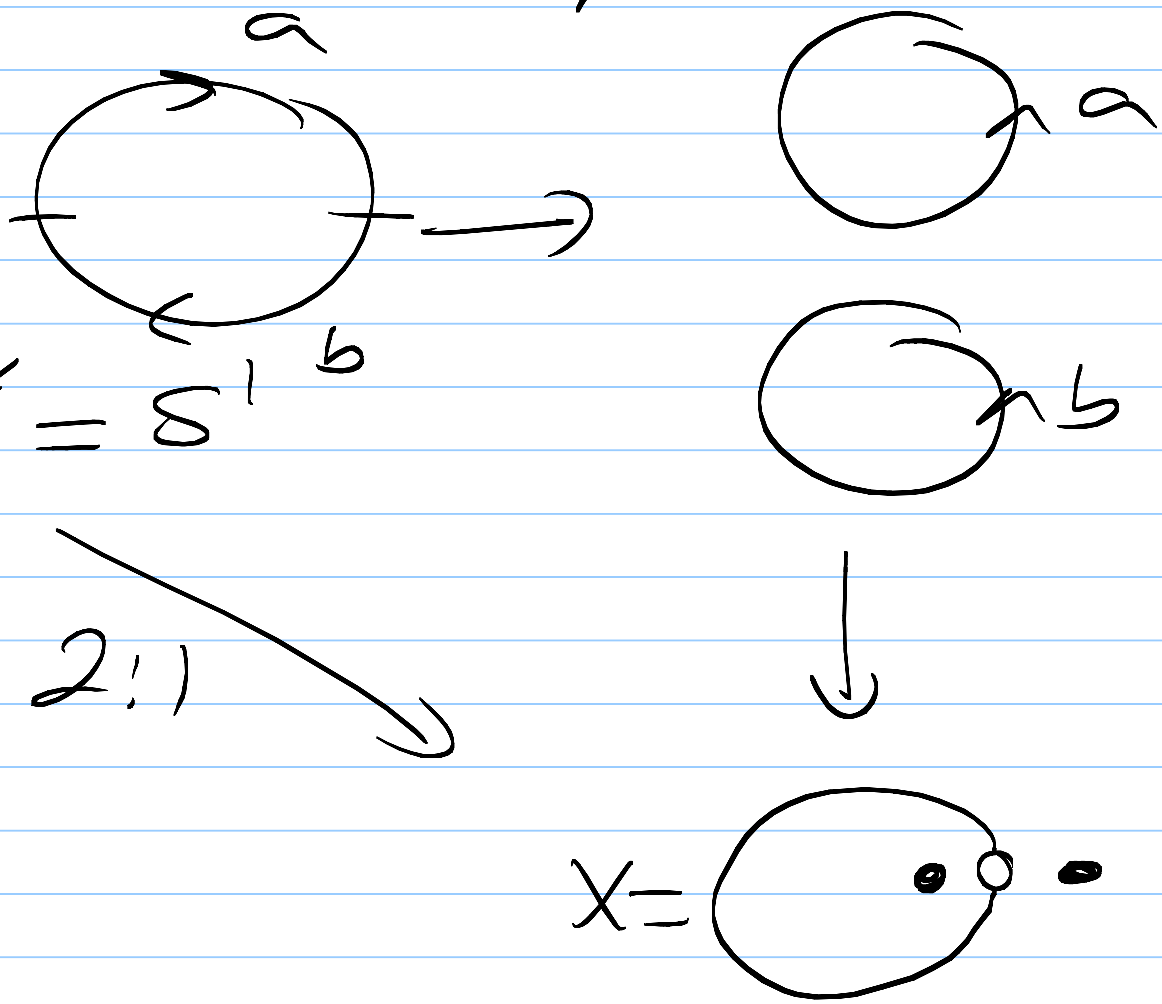
$\tilde{X}$:



Show that X deformation retracts onto

Then $\tilde{X} \longrightarrow X$: 

is a covering space.

k) 

$$X = S^1 \times \{0,1\} \big/ (x,0) \sim (x,1) \text{ if } x \neq 1$$

This is a 2 to 1 map and a local homeomorphism, but it is not a covering map.

Exercise: Prove the following:

Let $p: \tilde{X} \to X$ be a local homeomorphism, where $\tilde{X}$ and X are compact, connected Hausdorff spaces. Show that p is onto and a covering space.

Remark: A covering space $p: \tilde{X} \to X$ is called regular if $X = \tilde{X}/G$, where G is a group acting freely and properly discontinuously on $\tilde{X}$.

3) <u>Lifting Properties</u>:

Let $P: \tilde{X} \to X$ be a covering space and $f: Y \to X$ is a map. A lifting of $f: Y \to X$ is a map $\tilde{f}: Y \to \tilde{X}$ s.t. $f = p \circ \tilde{f}$:

$$\begin{tikzcd} & & \tilde{X} \arrow[d, "p"] \\ Y \arrow[urr, "f"] \arrow[rr, "f"'] & & X \end{tikzcd}$$

<u>Proposition</u>: (Homotopy Lifting)
Given a covering space $p: \tilde{X} \to X$, a homotopy $f_t: Y \to X$, and a map $\tilde{f}_0: Y \to \tilde{X}$ lifting of f_0, then there is a unique homotopy $\tilde{f}_t: Y \to \tilde{X}$ of $\tilde{f}_0$

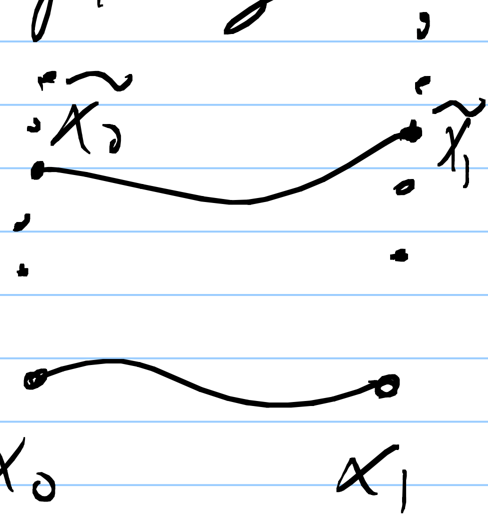
that lifts f_t.

Proof: Property (c) in the proof of $\pi_1(S^1)=\mathbb{Z}$ gives the proof. ∎

Special Cases

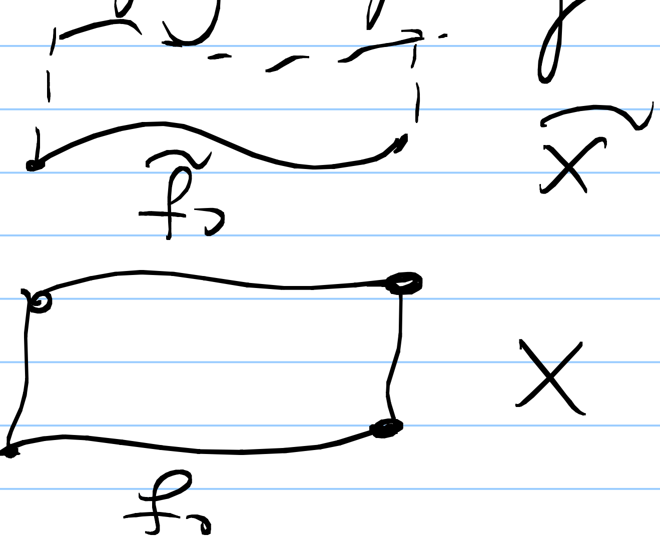
1) $Y=\{pt\}$ path lifting

$$\begin{tikzcd} & \tilde{X} \arrow[d] \\ Y\times I = I \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$



2) $Y=I$ homotopy lifting

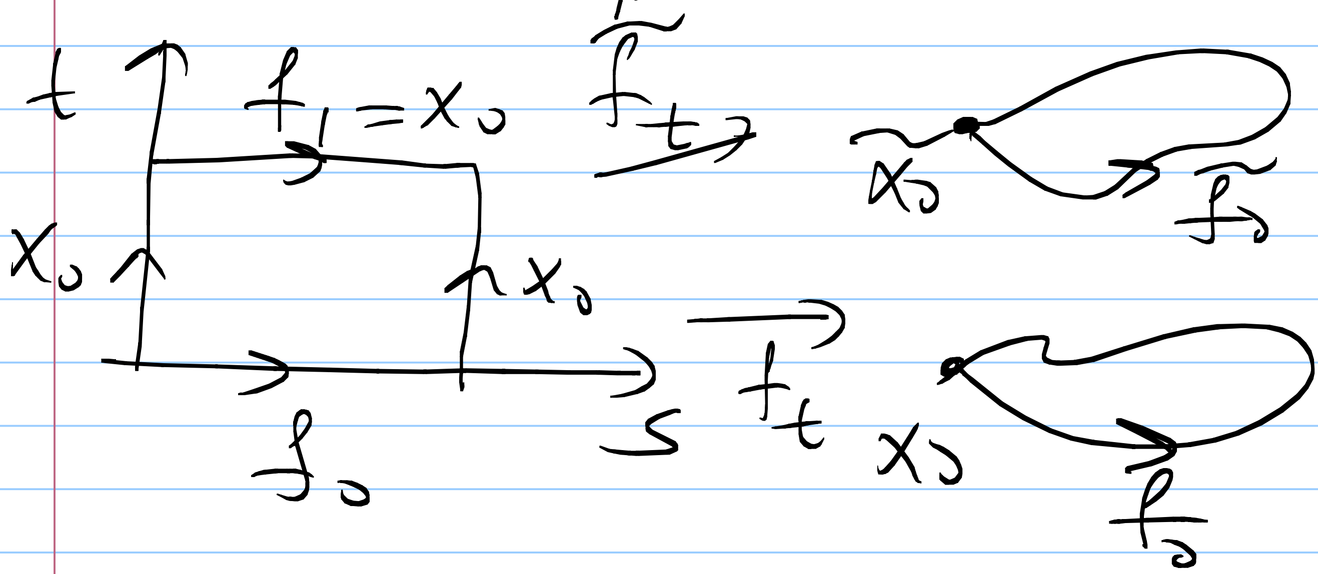
$Y\times I = I\times I$



Proposition: The map $p_{\#}: \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ induced by a covering projection is injective. The image subgroup $p_{\#}(\pi_1(\tilde{X}, \tilde{x}_0))$ consists of the homotopy classes of loops in X based at x_0 whose lifts to $\tilde{X}$ starting at $\tilde{x}_0$ are loops.

Proof: Let $\tilde{f}_0 : I \to \tilde{X}$ be a loop based at $\tilde{x}_0$ which represents a class in the kernel of the homomorphism $p_{\#}: \pi_1(\tilde{X}, x_0) \to \pi_1(X, x_0)$. Then $\tilde{f}_0$ is the unique lifting of $f_0 \doteq p \circ \tilde{f}_0 : I \to X$, a loop in X based at x_0. By assumption f_0 is homotopic to a constant. Hence there is a homotopy

$f_t : I \to X$ from f_0 to the constant loop at x_0.



By the previous proposition there is a homotopy $\tilde{f}_t$ of $\tilde{f}_0$ to $\tilde{f}_1$ so that

$$p \circ \tilde{f}_t = f_t \text{ for all } t.$$

In particular, $p \circ \tilde{f}_1(s) = f_1(s) = x_0$, for all $s \in [0,1]$. Hence, $\tilde{f}_1(s) \in p^{-1}(x_0)$ for all $s \in [0,1]$. Since $p^{-1}(x_0)$ is a discrete set and $\tilde{f}_1(0) = \tilde{x}_0$, $\tilde{f}_1(s) = \tilde{x}_0$ $\forall s \in [0,1]$

Hence, $\tilde{f}_1$ is a constant loop.
$\Rightarrow [\tilde{f_0}] = e$ in $\pi_1(\tilde{X}, \tilde{x_0})$.

The second statement is easy.

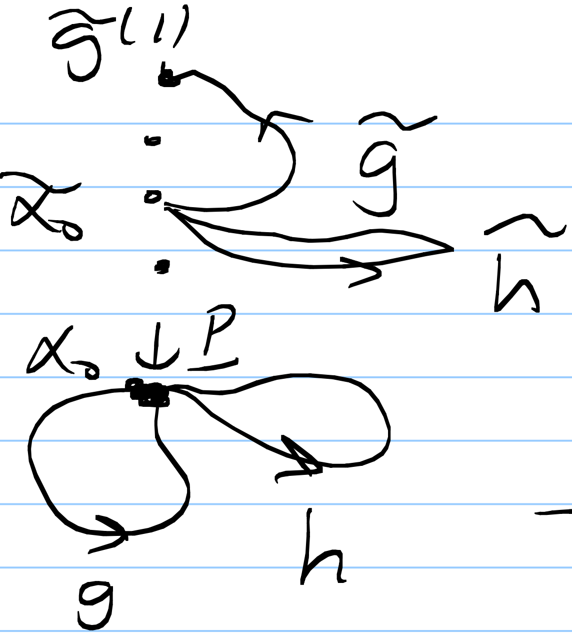
$\equiv$

<u>Remark</u>: If $p : \tilde{X} \to X$ is a covering map then the cardinality of $p^{-1}(x)$ is a locally constant function. Hence, if X is connected $|p^{-1}(x)|$ is independent of x. In this case it is called the degree of the covering: double covering, 3-fold, n-fold covering.
The degree of the covering is also said to be the number of sheets

of the covering.

<u>Proposition</u>: The number of sheets of a covering space $p:(\tilde{X},\tilde{x}_0)\to(X,x_0)$ of path connected spaces is equal to the index of $p_{\#}(\pi_1(\tilde{X},\tilde{x}_0))$ in $\pi_1(X,x_0)$.

<u>Proof</u>: Let g be a loop at x_0 and $\tilde{g}$ be its lift to $\tilde{X}$ starting at $\tilde{x}_0$. If $[h]\in H=p_*(\pi_1(\tilde{X},\tilde{x}_0))$ then the lift $\tilde{h}\tilde{g}$ of hg starting at $\tilde{x}_0$ has the same end point with $\tilde{g}$. So we get a well defined function $\Phi:\{H[g]\mid[g]\in\pi_1(X,x_0)\}\to p^{-1}(x_0)$ by sending the coset $H[g]$ to $\tilde{g}(1)$.



Since $\tilde{X}$ is path connected Φ is surjective. To see that Φ is injective assume that $\Phi(H[g_1]) = \Phi(H[g_2])$. Then $g_1 g_2^{-1}$ has a lift which is a loop at $\tilde{x}_0$. So $[g_1][g_2]^{-1} \in H$ and thus $H[g_1] = H[g_2]$. $\blacksquare$

Proposition (Lifting Criterion)

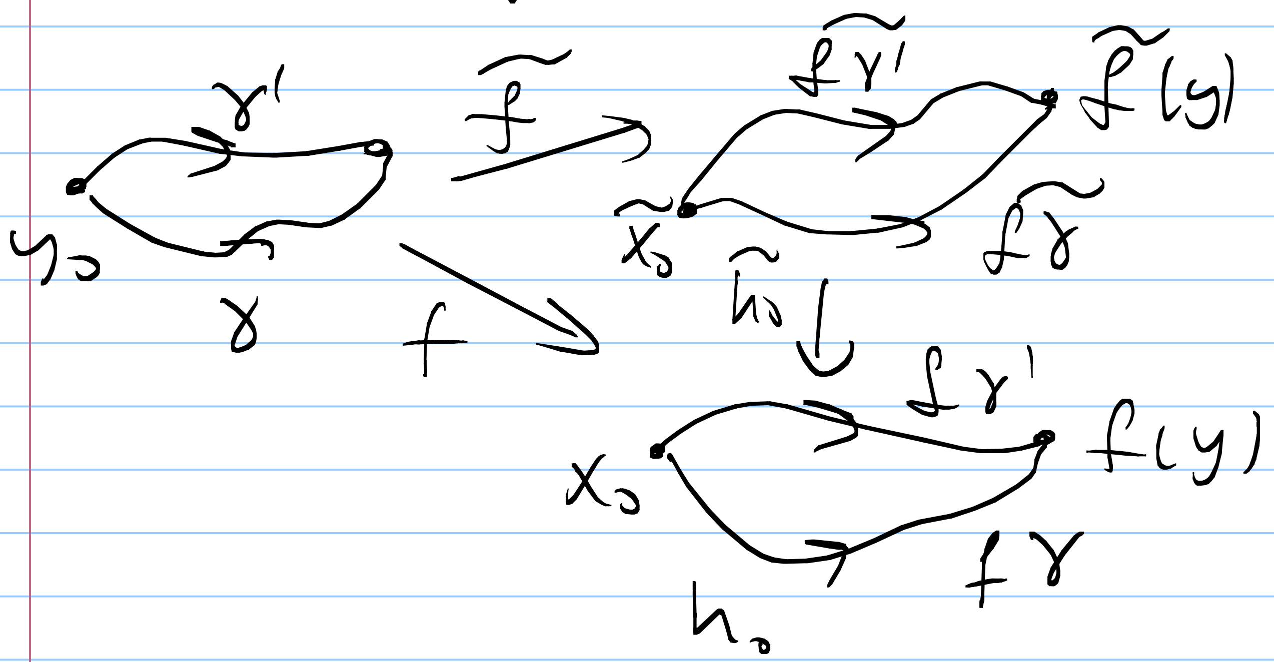
Let $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a covering space, $f : (Y, y_0) \to (X, x_0)$ a map where $f(y_0) = x_0$ and Y is path connected and locally path connected. Then f has a lift

$\tilde{f}: (Y, y_0) \to (\tilde{X}, \tilde{x}_0)$ if and only if
$f_{\#}(\pi_1(Y, y_0)) \subseteq p_{\#}(\pi_1(\tilde{X}, \tilde{x}_0))$.

<u>Proof</u> ($\Longrightarrow$) is clear since $f_{\#} = p_{\#} \circ \tilde{f}_{\#}$.
($\Longleftarrow$) Let $y \in Y$ and γ a path in Y from y_0 to y. The path $f\gamma$ in X starting at x_0 has a unique lift $\widetilde{f\gamma}$ starting at $\tilde{x}_0$. Define $\tilde{f}(y) = \widetilde{f\gamma}(1)$.

<u>Welldefinedness of $\tilde{f}$:</u>
Let γ' be another path from y_0 to y. Then $(f\gamma') \cdot (f\gamma)^{-1}$ is a loop h_0 at x_0 with $[h_0] \in f_{\#}(\pi_1(Y, y_0)) \subseteq p_{\#}(\pi_1(\tilde{X}, \tilde{x}_0))$. Hence, there is a homotopy h_t of h_0 to a loop h_1 that has a lift $\tilde{h}_1$ in $\tilde{X}$ based at $\tilde{x}_0$. By the homotopy

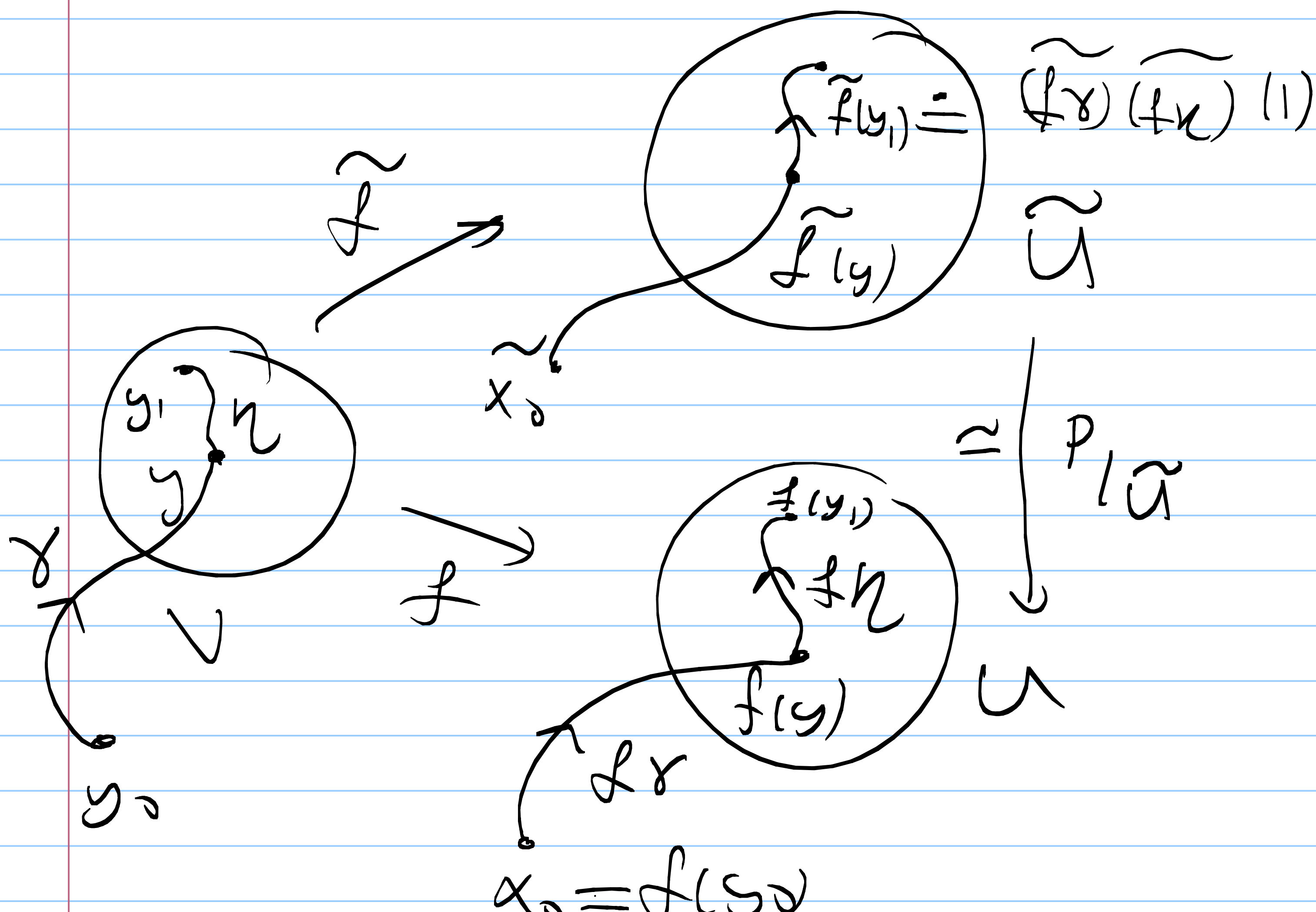
lifting h_t has a lift $\tilde{h}_t$. Since $\tilde{h}_1$ is a loop at $\tilde{x}_0$ so is $\tilde{h}_0$.



By the uniqueness of lifted paths, the first half of $\tilde{h}_0$ is $\widetilde{f\gamma'}$ and the second half is $(\widetilde{f\gamma})^{-1}$ with the common midpoint $\widetilde{f\gamma}(1) = \widetilde{f\gamma'}(1)$. This shows that $\tilde{f}$ is well defined.

$\tilde{f}$ <u>is continuous</u>: let $y \in Y$ and $U \subseteq X$ an open subset containing

$f(y)$ such that there is some $\tilde{U} \subseteq \tilde{X}$ with $p: \tilde{U} \to U$ is a homeomorphism. Choose a path connected open neighborhood V of y with $f(V) \subseteq U$. Fix a path γ from y_0 to y. The diagram below shows that $\tilde{f}(V) \subseteq \tilde{U}$ so that $\tilde{f}$ is continuous:



Proposition Given a covering space $p: \tilde{X} \to X$ and a map $f: Y \to X$ with two lifts $\tilde{f}_1$ and $\tilde{f}_2$ from Y to $\tilde{X}$ that agree at one point y_0 of Y, then if Y is connected these two lifts agree on all of Y.

Proof: Let A be the set of points in Y on which the two lifts agree. Since $p: \tilde{X} \to X$ is a local homeomorphism A is both and closed. On the other hand, A is nonempty since $y_0 \in A$. Finally, since A is connected $A = Y$ and thus the proof follows. $\blacksquare$

4) Classification of Covering Spaces:

First we will construct a universal covering space of a given space X provided that X is semilocally simply connected. In other words, every $x \in X$ has a neighborhood U such that

$$\pi_1(U, x) \longrightarrow \pi_1(X, x) \text{ is trivial.}$$

Example  is not semilocally simply connected.

Example: CW-complexes are locally contractible and thus they are locally simply connected $\Rightarrow$ semilocally simply connected.

Theorem: Let X be a path connected, locally path connected and semilocally simply connected. Then X has a universal covering $p: \tilde{X} \to X$, i.e., a simply connected covering space.

Idea of the proof:

Let $\tilde{X} = \{[\gamma] \mid \gamma$ is a path in X starting at $x_0\}$, where $[\gamma]$ denote the homotopy class of γ with respect to homotopies fixing the endpoints $\gamma(0)$ and $\gamma(1)$.

Also define
$$p: \tilde{X} \longrightarrow X$$
$$[\gamma] \longmapsto \gamma(1)$$

Clearly, p is surjective.

Next, let $\mathcal{U}$ denote the collection of path connected open sets $U \subseteq X$

s.t. $\pi_1(U) \to \pi_1(X)$ is trivial.

Note that if $V \subseteq U \in \mathcal{U}$ then $\pi_1(V) \to \pi_1(U) \to \pi_1(X)$ is also trivial and thus $V \subseteq \mathcal{U}$.

Hence, $\mathcal{U}$ is a basis for the topology on X.

Now given a set $U \in \mathcal{U}$ and a path γ in X from x_0 to a point in U, let

$$U_{[\gamma]} = \{[\gamma \cdot \eta] \mid \eta \text{ is a path in } U \text{ with } \eta(0) = \gamma(1)\}.$$

- $U_{[\gamma]}$ depends only on $[\gamma]$.
- Since U is path connected $p: U_{[\gamma]} \to U$ is onto
- p is also injective.

This because η and η' are homotopic in X via homotopy fixing the end points, because $\pi_1(U) \to \pi_1(X)$ is trivial.
Thus $[\gamma \cdot \eta] = [\gamma \eta']$.

[<u>Conclusion</u> $p: U_{[\gamma]} \to U$ is a bijection. Using p we can carry the topology on U to $U_{[\gamma]}$ making $p|_{U[\gamma]}$ a homeomorphism.]

- The sets $U_{[\gamma]}$ form a basis for a topology on $\tilde{X}$ so that each $p|: U_{[\gamma]} \to U$ is a homeo morphism.

- $p : \tilde{X} \to X$ is a continuous map.
- $p : \tilde{X} \to X$ is a covering projection.
- $\tilde{X}$ is simply connected.

Theorem (Classification)

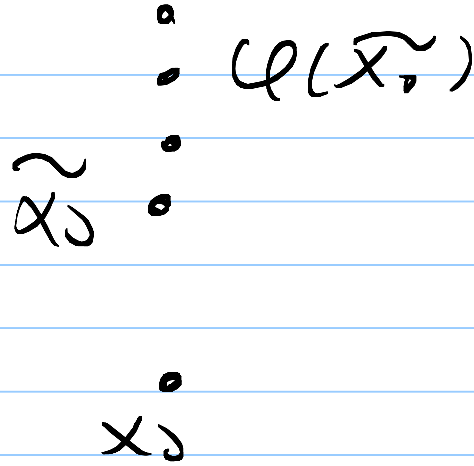
Suppose X is a path connected, locally path connected and semilocally simply connected. Then there is a bijection between the set of base point preserving isomorphism classes of path connected covering spaces $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ and the set of subgroups of $\pi_1(X, x_0)$

associating the covering space $(\tilde{X}, \tilde{x}_0)$ to $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$. If base points are ignored, this correspondence gives a bijection between the isomorphism classes of covering spaces and conjugacy classes of subgroups of $\pi_1(X, x_0)$.

5) Deck Transformations and Group Actions

Given a covering space $p: \tilde{X} \to X$ of path connected spaces its isomorphisms are called Deck transformations. Clearly they form a group denoted $G(\tilde{X})$.

$$\begin{tikzcd} \tilde{X} \arrow[rr, "\varphi"] \arrow[dr, "p"'] & & \tilde{X} \arrow[dl, "p"] \\ & X & \end{tikzcd}$$



The unique lifting property implies that φ is determined by its image at a single point. Hence, $|G(\tilde{X})| \leq |p^{-1}(x_0)|$ = the number of sheets of the covering.

<u>Defn</u> A covering $p: \tilde{X} \to X$ is called normal if $p_\#(\pi_1(\tilde{X}, \tilde{x}_0)) \leq \pi_1(X, x_0)$ is a normal subgroup.

<u>Proposition</u> Given a covering of path connected spaces $p: \tilde{X} \to X$ the following are equivalent:

a) $p: \tilde{X} \to X$ is normal
b) For any $\tilde{x}$ and $\tilde{x}'$ in $\tilde{X}$ with $p(\tilde{x}) = p(\tilde{x}')$ there is a deck transformation $\varphi \in G(\tilde{X})$ s.t. $\varphi(\tilde{x}) = \tilde{x}'$.
c) For any $[\gamma] \in \pi_1(X, x_0)$ either all lifts of γ are loops or all lifts of γ are non loops.
Moreover, in this case, $\tilde{X} = X / G(\tilde{X})$ and $\pi_1(X) / p_{\#}(\pi_1(\tilde{X})) \simeq G(\tilde{X})$.

Proposition: For a covering spaces of path connected spaces $G(\tilde{X})$ is isomorphic to $N(H)/H$ where $H = p_{\#}(\pi_1(\tilde{X}, \tilde{x}_0))$ and $N(H)$ is its normalizer.

6) Applications to Group Theory

Construct subgroups of free groups of a given index. See examples at page 57.

Exercise 1) Find a normal subgroup N of F_2 such that F_2/N is isomorphic to the group of isometries of a cube/tetrahedron or any other platonic solid.

2) If $F_n \leq F_m$ then

then $m-1 \mid n-1$ and $[F_n : F_m] = \dfrac{n-1}{m-1}$.

7) Branched Coverings of Surfaces

Let G be a finite group acting freely on a simplicial complex K via simplicial homeomorphisms. Then the quotient space $X = |K|/G$ has a cell structure, say $L = K/G$.

Proposition If K, G and L are as above then $\chi(K) = |G|\,\chi(L)$.

Now suppose that there is a finite set of vertices $\{x_1, \dots, x_N\}$ in K so that the action of G on $X - \{x_1, \dots x_N\}$ is free. Suppose that $\{x_1, \dots, x_n\} = O_1 \cup \dots \cup O_n$ where each $O_i = Gx_j$ for some $j \in \{1, \dots, N\}$, and $O_i \cap O_j = \emptyset$ if $i \neq j$.

Then the same argument shows that $\chi(K) - N = |G| (\chi(L) - n)$

This is known as Riemann–Hurwitz Theorem in Algebraic Geometry.

In this case, we say that $K \to L$ is a branched cover with branch locus $\{x_1 \ldots x_n\}$.

Theorem (Riemann)

If G is a finite group acting on a finite 2-dim'l simplicial complex K so that $|K| = \Sigma_g$ and the action is free outside finitely many vertex. Then $|G| \leq 84(g-1)$ provided that $g \geq 2$.

Proof: $C = \{ p \in \Sigma_g \mid |\mathrm{Orb}_G(p)| < |G| \}$

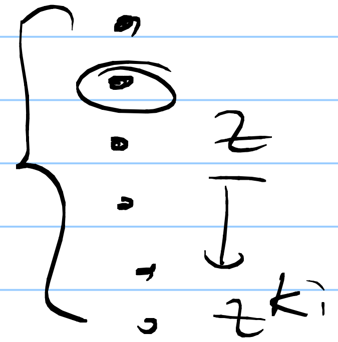
i.e. $\mathrm{Stab}_G(p) \neq \mathrm{id}$.

Then G acts freely on $\Sigma_g - C$.

By assumption C is a finite set and G acts on C also.

$$C = O_1 \cup O_2 \cup \cdots \cup O_n$$

$$\begin{array}{c} \Sigma_g \\ \downarrow \\ \Sigma_h = \Sigma_g / G \end{array}$$

O_1 

$$k_i = \frac{|G|}{|O_i|}$$

Let $N = |O_1| + \cdots + |O_n| = |C|$

Then by the Riemann-Hurwitz theorem

$$2 - 2g - N = |G|(2 - 2h - n)$$

$$\Rightarrow 2-2g - \sum_{i=1}^{n} |O_i| = |G| \left(2-2h - \sum_{i=1}^{n} 1\right)$$

$$\Rightarrow 2-2g - \sum_{i=1}^{n} \frac{|G|}{k_i} = |G| \left(2-2h - \sum_{i=1}^{n} 1\right)$$

$$\Rightarrow 2(1-g) = |G| \left(2-2h - \sum_{i=1}^{n} (1 - 1/k_i)\right)$$

$$\Rightarrow 2(g-1) = |G| \left(2(h-1) + \sum_{i=1}^{n} (1 - 1/k_i)\right)$$

$$|G| = \frac{2(g-1)}{2(h-1) + \sum_{i=1}^{n} (1 - 1/k_i)}$$

Here $h \geq 0$ and all $k_i \geq 2$. So to find an upper bound for $|G|$ we need to minimize

$$2(h-1) + \sum_{i=1}^{n} (1 - 1/k_i).$$

Smallest value is obtained for $n=4$ is obtained at

$n=0$, $k_1=2$, $k_2=2$, $k_3=2$, $k_4=3$ which gives

$$-2+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{2}{3}=\frac{1}{6}$$

$$\Rightarrow |G| \leq 12(g-1).$$

and for $0<n\leq 3$ at $h=0$, $n=3$, $k_1=2$, $k_2=3$, $k_3=7$ which gives

$$-2+\frac{1}{2}+\frac{2}{3}+\frac{6}{7}=\frac{1}{42}$$

$$\Rightarrow |G| \leq 42\cdot 2(g-1)=84(g-1).$$

<u>Remark</u> Indeed one can show that if there are more that $n\geq 4$ singular fibers then $|G|\leq 12(g-1)$.

Proposition: Under the assumptions of Riemann-Hurwitz theorem $g(K) \geq g(L)$.

Proof: We have as before $2-2g-N = |G|(2-2h-n)$. Also note that $|G|n \geq N$ (In fact $|G|n > N$ if $N>0$). Then $2-2g = (2-2h)|G| + N - n|G|$.

$\Rightarrow 2-2g \leq (2-2h)|G|$.

Since $|G| \geq 2 \Rightarrow$

$g-1 \geq 2(h-1) = 2h-2$

$\Rightarrow g-h \geq h-1$.

If $h \geq 1$ then $g \geq h$ and

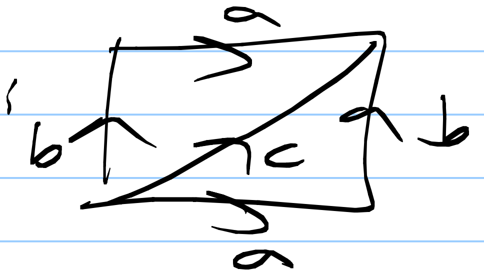
if $h=0$ then $g \geq 0 = h$. $\blacksquare$

HOMOLOGY

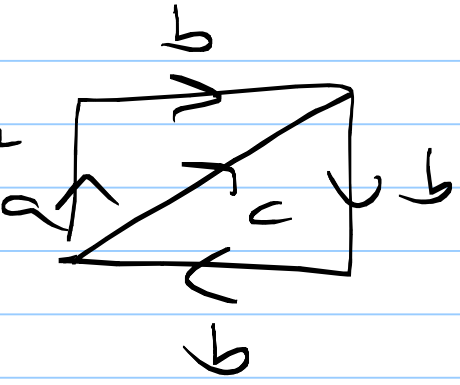
1) Simplicial Homology

Δ-complex

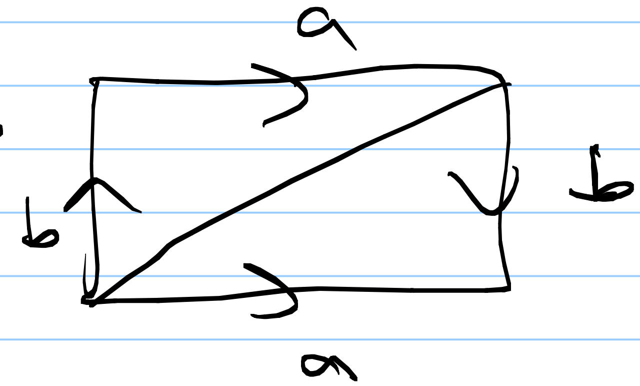
T^2:



$\mathbb{RP}^2$



KB:



Recall that

$$\Delta^n = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid t_i \geq 0, \sum_{i=0}^{n} t_i = 1\}$$

is the standard n-simplex.

More generally, if $\{v_0, v_1, \dots, v_n\}$ is a set of vectors in $\mathbb{R}^m$ such that $\{v_1 - v_0, \dots, v_n - v_0\}$ is a linearly independent set then the n-simplex determined by $\{v_0, \dots, v_n\}$ is defined by

$$[v_0, \dots, v_n] \doteq \left\{ \sum_{i=0}^{n} t_i v_i \;\middle|\; t_i \geq 0, \sum_{i=0}^{n} t_i = 1 \right\}.$$

Note that

$$\Delta^n \longrightarrow [v_0, \dots, v_n]$$

$$(t_0, \dots, t_n) \longmapsto \sum_{i=0}^{n} t_i v_i \quad \text{is a}$$

homeomorphism. Any simplex $[v_{i_1}, \dots, v_{i_k}]$, $i_1, \dots, i_k \in \{0, 1, \dots, n\}$ is called a face on $[v_0, \dots, v_n]$.

A delta complex is a quotient space of some disjoint union of simplices, where certain faces of simplices are identified by linear isomorphisms. Note that Δ-complexes are naturally

CW-complexes.

<u>Simplicial Homology</u>: Let X be a Δ-complex. Define

$\Delta_n(X)$ = free abelian group with basis the open n-simplices e^n_α of X.

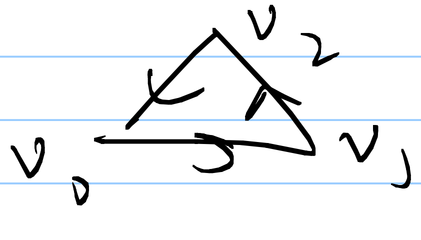
Elements of $\Delta_n(X)$ are called n-chains: $\sum_\alpha n_\alpha e^n_\alpha$, $n_\alpha \in \mathbb{Z}$, $n_\alpha = 0$ for all but finitely many α.

<u>Boundary of a simplex</u>

$$\partial([v_0, \dots, v_n]) = \sum_{i=0}^{n} (-1)^i [v_0, \dots, \hat{v}_i, \dots, v_n].$$

<u>Ex</u> $\partial[v_0, v_1] = [v_1] - [v_0]$

$$\partial[v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1].$$



Lemma $\partial_n \circ \partial_{n-1} = 0$.

Remark $\partial_n \circ \partial_{n-1} = 0 \iff$
$\operatorname{Im} \partial_{n-1} \subseteq \ker \partial_n$.

Definition The nth simplicial homology of a Δ-complex X is defined to be the quotient group $H_n^\Delta(X) = \dfrac{\ker \partial_n}{\operatorname{Im} \partial_{n-1}}$

Example $S^1, T^2, S^n, \mathbb{RP}^2$.

2) **Singular Homology** Consider maps $\sigma : \Delta^n \to X$ (continuous)
$C_n(X) =$ the free abelian group with basis $\{\sigma : \Delta^n \to X \mid \sigma$ is continuous$\}$.

$$\partial_n : C_n(X) \longrightarrow C_{n-1}(X)$$

$$\sum n_\alpha \sigma_\alpha \longmapsto \sum n_\alpha \partial \sigma_\alpha$$

where if $\sigma_\alpha : [v_0, v_1, \dots, v_n] \to X$

then $\partial \sigma_\alpha = \sum_{i=0}^{n} (-1)^i \sigma_\alpha |_{[v_0, \dots, \hat{v}_i, \dots, v_n]}$.

Clearly, $\partial_{n+1} \circ \partial_n = 0$ and thus the nth singular homology of X is defined by

$$H_n(X) = \frac{\ker \partial_n}{\operatorname{Im} \partial_{n+1}}.$$

Remark Compare the two homology theories.

Proposition: If X is nonempty and path connected then $H_0(X) = \mathbb{Z}$.

Hence if $X = \bigcup_\alpha X_\alpha$, where

each X_α is path connected, then
$H_0(X) \simeq \bigoplus_\alpha \mathbb{Z}$.

3) If $f: X \to Y$ is a continuous map then $f_\#: C_n(X) \to C_n(Y)$ by
$f_\#(\sigma) = f \circ \sigma : \Delta^n \to Y$ and then

$$f_\#\Big(\sum_i n_i \sigma_i\Big) = \sum_i n_i f_\#(\sigma_i)$$
$$= \sum_i n_i (f \circ \sigma_i).$$

Note that $f_\#(\partial \sigma) = \partial f_\#(\sigma)$

and thus we get a commutative diagram of chains complexes:

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1}(X) \arrow[r, "\partial"] \arrow[d, "f_\#"] & C_n(X) \arrow[r, "\partial"] \arrow[d, "f_\#"] & C_{n-1}(X) \arrow[r] \arrow[d, "f_\#"] & \cdots \\
\cdots \arrow[r] & C_{n+1}(Y) \arrow[r] & C_n(Y) \arrow[r] & C_{n-1}(Y) \arrow[r] & \cdots
\end{tikzcd}$$

$\Rightarrow f_* : H_n(X) \longrightarrow H_n(Y)$.

Note that

i) $(fg)_* = f_* g_*$

ii) $(id_X)_* = id_{H_n(X)}$, for all n.

<u>Theorem</u>: If two maps $f, g : X \to Y$ are homotopic then $f_* = g_*$.

<u>Corollary</u>: If $f : X \to Y$ is a homotopy equivalence then $f_* : H_n(X) \to H_n(Y)$ is an isomorphism.

<u>Definition</u> (Reduced homology)

$$\cdots \to C_n(X) \to C_{n-1}(X) \to \cdots$$

$$\to C_1(X) \to C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

$\epsilon : C_0(X) \to \mathbb{Z}$ by

$\epsilon(\sum n\sigma_\lambda) = \sum n$.

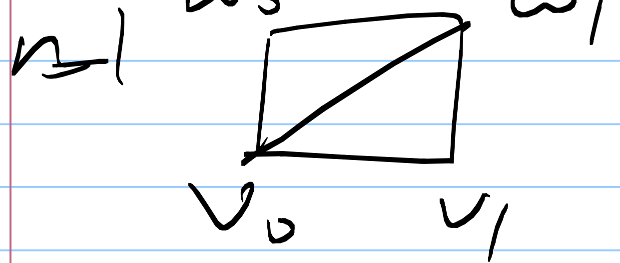
Its homology is called the reduced homology of X and denoted by $\tilde{H}_n(X)$.

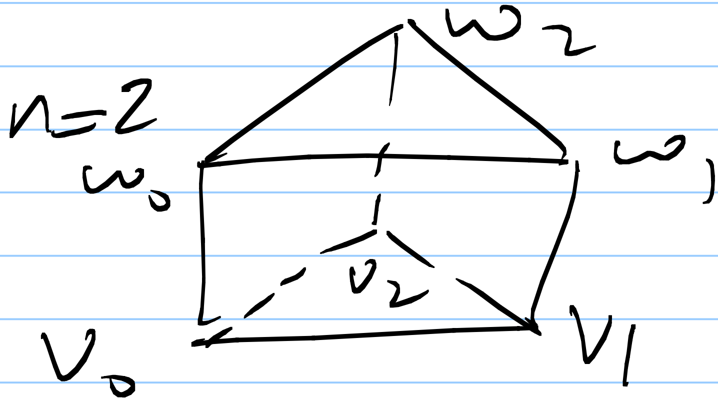
Proposition $\tilde{H}_n(X) = H_n(X)$ if $n>0$ and $\tilde{H}_0(X) \oplus \mathbb{Z} \cong H_0(X)$.

Proposition If X is contractible then $\tilde{H}_n(X) = 0$ for all n.

Idea of the proof of the Theorem,

Subdivide $\Delta^n \times I$ into $(n+1)$-simplices





$$\Delta^n \times I = \bigcup_{i=0}^{n} [v_0, \dots, v_i, w_i, \dots, w_n]$$

$$\Delta^n \times \{0\} = [v_0, \dots, v_n]$$
$$\Delta^n \times \{1\} = [w_0, \dots, w_n]$$

Given a homotopy $F: X \times I \to Y$ define the Prism operator

$$P: C_n(X) \to C_{n+1}(Y)$$

<u>Claim</u> $\partial P = g_\# - f_\# - P\partial$

where $f(x) = F(x,0)$, $g(x) = F(x,1)$.

<u>Claim</u> proves the theorem:
For any cycle $\alpha \in C_n(X)$,

$$g_\#(\alpha) - f_\#(\alpha) = \partial P(\alpha) + P\partial(\alpha)$$
$$= \partial P(\alpha)$$

$\Rightarrow [g_\#(\alpha)] = [f_\#(\alpha)]$

$\Rightarrow g_*(\alpha) = f_*(\alpha)$

$\equiv$

4) Exact Sequences and Excision:

X space, $A \subseteq X$ subspace.

We'll relate $H_*(X)$, $H_*(A)$ and $H_*(X/A)$.

Chain complex:

$$\to A_{n+1} \xrightarrow{\partial_{n+1}} A_n \xrightarrow{\partial_n} A_{n-1} \to$$

$\partial_n \circ \partial_{n+1} = 0$ or equivalently $\operatorname{Im} \partial_{n+1} \subseteq \ker \partial_n$.

It is called exact if

$$\operatorname{Im} \partial_{n+1} = \ker \partial_n.$$

Remark

1) $0 \to A \xrightarrow{\alpha} B$ is exact iff

$\ker\alpha = 0 \iff \alpha$ is injective.

2) $A \xrightarrow{\alpha} B \to 0$ is exact iff $\operatorname{Im}\alpha = B \iff \alpha$ is surjective.

3) $0 \to A \xrightarrow{\alpha} B \to 0$ is exact iff α is an isomorphism

4) $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is exact iff α is injective, β is surjective and $\operatorname{Im}\alpha = \ker\beta$.

<u>Theorem</u> If X is a space and A is a nonempty closed subset that is deformation retract of some neighborhood in X, then there is an exact sequence

$$\cdots \to \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \xrightarrow{j_*} \tilde{H}_n(X/A) \xrightarrow{\partial} \tilde{H}_{n-1}(A) \to$$

where $i: A \hookrightarrow X$ is the inclusion, $j: X \to X/A$ is the quotient map and ∂ is the boundary homomorphism.

<u>Remark</u> If X is CW complex and $A \subseteq X$ is subcomplex then (X,A) is a such pair, called a good pair.

<u>Corollary</u> $\tilde{H}_n(S^n) \simeq \mathbb{Z}$ and $\tilde{H}_i(S^n) = 0$ if $i \neq n$.

<u>Corollary</u> ∂D^n is <u>not</u> a retract of D^n and thus any map $f: D^n \to D^n$ has a fixed point.

Idea of proof $A \subseteq X$

Define $C_n(X,A) \doteq \dfrac{C_n(X)}{C_n(A)}$.

Note that $\partial : C_n(X) \to C_{n-1}(X)$ induces a homomorphism

$\partial : C_n(X,A) \to C_{n-1}(X,A)$.

Clearly $\partial^2 = 0$ in

$$\to C_{n+1}(X,A) \xrightarrow{\partial_{n+1}} C_n(X,A) \xrightarrow{\partial_n} C_{n-1}(X,A)$$

and thus we may define

$$H_n(X,A) = \frac{\ker \partial_n}{\operatorname{Im} \partial_{n+1}}$$

Lemma If $0 \to A_\alpha \to B_\alpha \to C_\alpha \to 0$ is an exact sequence of chain complexes then there is a long exact sequence in homology

$$\to H_n(A) \to H_n(B) \to H_n(C) \xrightarrow{\partial} H_{n-1}(A)$$

<u>Remark</u>

$$0 \to C_*(A) \to C_*(X) \to C_*(X,A) \to 0$$

$$\Rightarrow \cdots \to H_n(A) \to H_n(X) \to H_n(X,A) \to H_{n-1}(A)$$

<u>Remark</u> $B \subseteq A \subseteq X$

$$0 \to C_n(A,B) \to C_n(X,B) \to C_n(X,A) \to 0$$

$$\Rightarrow \cdots \to H_n(A,B) \to H_n(X,B) \to H_n(X,A)$$
$$\to H_{n-1}(A,B) \to \cdots$$

Theorem (Excision)

Let $Z \subseteq A \subset X$ subspaces such that $\overline{Z} \subset \mathrm{Int}(A)$. Then the inclusion $(X-Z, A-Z) \hookrightarrow (X,A)$ induces isomorphism

$$H_n(X-Z, A-Z) \xrightarrow{\sim} H_n(X,A), \ \forall n.$$

Remark Note that

$Z \subseteq A \subseteq X$, $\bar{Z} \subseteq \mathrm{Int}(A)$

implies $X = \mathrm{Int}(A) \cup \mathrm{Int}(B)$

where $B = X - Z$.

Idea of the proof uses

$\boxed{C_n(X) \simeq C_n(A+B)}$ critical point.

$$\Rightarrow \frac{C_n(X)}{C_n(A)} \simeq \frac{C_n(A+B)}{C_n(A)}$$

$$\Rightarrow H_n(X,A) \simeq H(\quad)$$

Also $\dfrac{C_n(B)}{C_n(A\cap B)} \longrightarrow \dfrac{C_n(A+B)}{C_n(A)}$

is also an isomorphism since both quotient groups are free with basis the singular

n-simplices in B that do not lie in A. Then

$$\begin{array}{c} H_n(B, A\cap B) \simeq H_n(X,A) \\ \| \\ H_n(X-Z, A-Z) \end{array}$$

$\blacksquare$

<u>Proposition</u> For good pairs (X,A) the quotient map $q:(X,A)\longrightarrow(X/A, A/A)$ induces isomorphisms

$$q_*: H_n(X,A)\longrightarrow H_n(X/A, A/A)\simeq \tilde{H}_n(X/A) \quad \forall n.$$

5) Singular Homology is equal to the simplicial Homology. For a simplicial complex X

$H_n^{\Delta}(X) \simeq H_n(X)$.

6) Betti numbers and Euler Characteristic.