Recent Developments in Financial Mathematics

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Agenda

Bachelor Studies: Probability Theory

Figure: Professor Azize Hayfavi

Master Studies: Stochastic Processes, Interest Rate Modelling

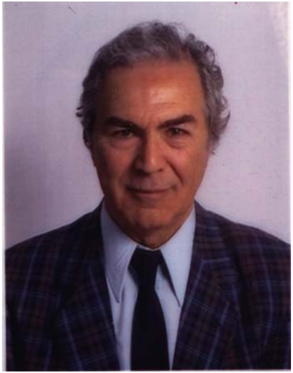


Figure: Professor Hayri Körezlioğlu

Ph.D. Studies: Stochastic Differential Equations, Malliavin Calculus

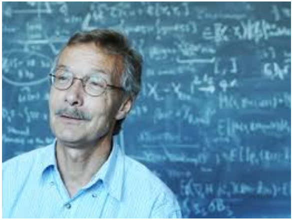


Figure: Professor Bernt Øksendal

Ph.D. Studies: Lévy Processes, Stochastic Analysis

Figure: Professor Giulia Di Nunno

Ph.D. Studies: Malliavin Calculus

Figure: Professor Paul Malliavin

Observations:

- Stock prices move randomly according to efficient market hypothesis.
- Absolute change in the price of a stock is not a useful quantity. Increase of one unit in a stock worth 10 units is much more significant than the one with a stock worth 100 units.
- Hence, relative change must be modeled: dS/S
- $dS/S = (\mu + \sigma noise)dt$ (more realistic, consider some market random effects)
- noise (white noise)= $\frac{dB(t)}{dt}$

Historical Data of S&P 500 Stock Index

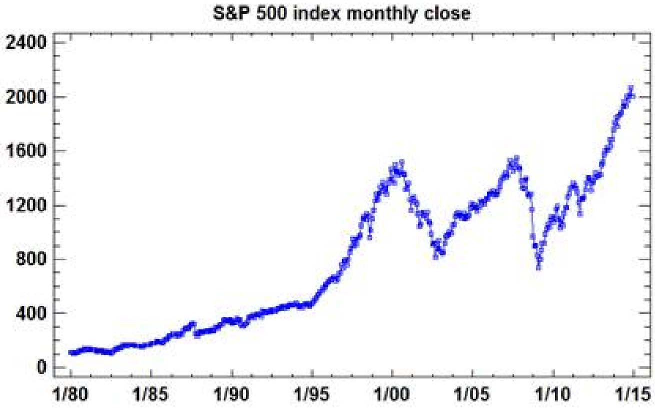


Figure: monthly closing values of the S&P 500 from January 1980 to November 2014 (shows consistent exponential growth)

Random variable

$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$

Definition

A real valued random variable, X, is a real valued $\mathcal{F}$-measurable function defined on Ω. *i.e.*

$$X : \Omega \to \mathbb{R}$$

$$(\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{R}, \mathcal{B}, \mathbb{P}_X)$$

$B = (-\infty, x]$, $x \in \mathbb{R}$,

$$\begin{aligned} \mathbb{P}_X((-\infty, x]) &= \mathbb{P}(\{\omega : X(\omega) \in (-\infty, x]\}) \\ &= \mathbb{P}(\{\omega : X(\omega) \leq x\}) = F_X(x) \end{aligned}$$

Stochastic Process

Definition

A real valued stochastic process is a sequence of real random variables, *i.e.*

$$(t, \omega) \to X_t(\omega) : I \times \Omega \to \mathbb{R}$$

is a *stochastic process* which is a parameterized collection of real-valued rvs $\{X_t\}_{t \in I}$.

Brownian Motion

Definition

A Brownian is a real valued, continuous stochastic process $X(t, \omega)$, $t \geq 0, \omega \in \Omega$, with independent and stationary increments, i.e.,

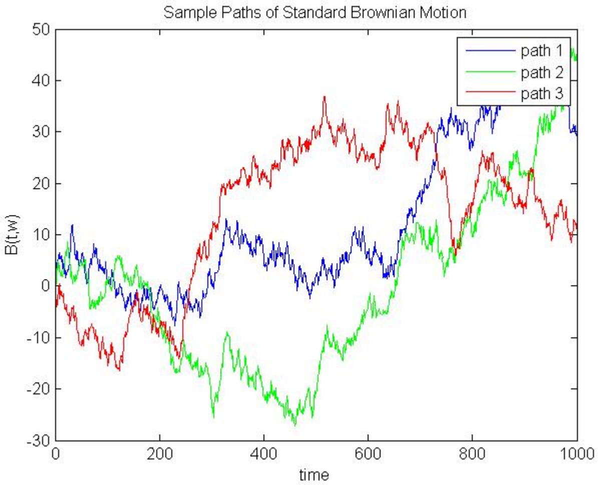
- continuity: The map $t \to X(t, \omega)$ is continous.
- independent increments: $\forall s \leq t$, $X(t) - X(s)$ is indepedent of $\mathcal{F}_s = \sigma(X(u), u \leq s)$.
- stationary increments: $\forall s \leq t$, $h > 0$, $X(t + h) - X(s + h)$ have the same probability law with $X(t) - X(s)$.

Standard Brownian Motion

A standard Brownian motion, $B(t, \omega)$, $t \in [0, T]$, $\omega \in \Omega$, is a Brownian motion with the following properties:

- $B(0) = 0 \ \mathbb{P} - a.s.$
- $\mathbb{E}[B(t)] = 0$
- $\mathbb{E}[B(t)^2] = t$

Simulation of Brownian Motion



Properties of Brownian Motion

- cont. everywhere but nowhere differentiable,
- hits any and every real value no matter how large or small it is,
- It has $1/2$-self similarity, *i.e.* $B(at) = a^{1/2}B(t)$.

p-Variation

$$f : [a, b] \to \mathbb{R}$$

$$\triangle_n := \{a = t_0 < t_1 < \ldots t_{n-1} < t_n = b\}$$

$$||\triangle_n|| := max_{1 \leq i \leq n}(t_i - t_{i-1})$$

$$Q_p(f; a, b, \triangle_n) := \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|^p$$

Definition

The *pth variation* of f on $[a, b]$ is defined as follows:

$$V_p(f; a, b) := sup_{\triangle_n} Q_p(f; a, b, \triangle_n)$$

where supremum is taken over all partitions of $[a, b]$.

Variation of Brownian Motion

Taylor and Duke (1972) showed that

$$V_p(B; a, b) < \infty \iff p > 2$$

For $p = 2$, instead of considering the supremum over all partitions, if we restrict ourselves to the sequences of partitions $\{\triangle_n\}$ for which $||\triangle_n|| \to 0$, then we obtain the following result:

Theorem

$$Q_2(B; a, b, \triangle_n) \to b - a \quad in\, L^2$$

as $||\triangle_n|| \to 0$. *Moreover, the quadratic variation of a Brownian motion* B *on the interval* $[a, b]$ *is the limit of* $Q_2(B; a, b, \triangle_n)$ *and denoted by* $< B, B >_{[a,b]}$.

Stochastic Integration

$H(t,\omega) = \sum_{i=1}^{p} \phi_i(\omega)\mathcal{I}_{(t_{i-1},t_i]}(t)$
where ϕ is $\mathcal{F}_{t_{i-1}}$-measurable and bounded,
$0 = t_0 < t_1 < \ldots < t_p = T$.

$$I(H)(t) := \sum_{1 \le i \le k} \phi_i(B(t_i) - B(t_{i-1})) + \phi_{k+1}(B(t) - B(t_k))$$

for any $t \in (t_k, t_{k+1}]$.

Itô Process

$X(t,\omega), t \in [0,T]$ is called Itô process if it can be written as

$$X(t) = X(0) + \int_0^t K(s)ds + \int_0^t H(s)dB(s),$$

where

- $X(0)$ is $\mathcal{F}_0$-measurable.
- $\int_0^T |K(s)|ds < \infty \qquad \mathbb{P} - a.s.$
- $\int_0^T |H(s)|^2 ds < \infty \qquad \mathbb{P} - a.s.$

Stochastic Differential Equations

More generally, consider the following equation:

$$dX(t) = b(t, X(t))dt + \sigma(t, X(t))dB(t) \quad (1)$$

where $X(0) = x \in \mathbb{R}$,
$b : [0, T] \times \mathbb{R} \to \mathbb{R}$,
$\sigma : [0, T] \times \mathbb{R} \to \mathbb{R}$

Questions Come to Our Minds at First Glance

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   1. b and σ satisfy Lipschitz and polynomial growth condition and $\mathbb{E}[|x|^2] < \infty$
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4. $\mathcal{E} = \{(X_s)_{0\leq s\leq T}, \mathcal{F}_t -$measurable continuous processes s.t. $\mathbb{E}(\sup_{s\leq T} |X_s|^2) < \infty\}$

Itô Formula

Let $X(t, \omega)$, $t \in [0, T]$ be an Itô process

$$X(t) = X(0) + \int_0^t K(s)ds + \int_0^t H(s)dB(s),$$

and $f \in C^{1,2}$, then

$$\begin{aligned} f(t, X(t)) &= f(0, X(0)) + \int_0^t \frac{\partial f}{\partial s}(s, X(s))ds + \int_0^t \frac{\partial f}{\partial x}(s, X(s))dX(s) \\ &+ \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s, X(s))d < X, X >_s \end{aligned}$$

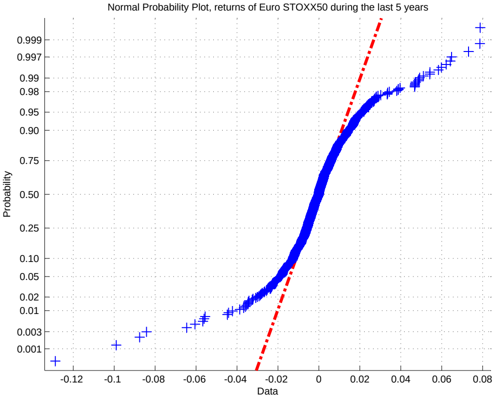
Itô Formula in Practice

$f(t,x) = x^2$, $X(t) = B(t)$;

$$B(t)^2 = t + 2\int_0^t B(s)dB(s)$$

Lévy processes in Financial Mathematics

- In reality returns are not normally distributed!



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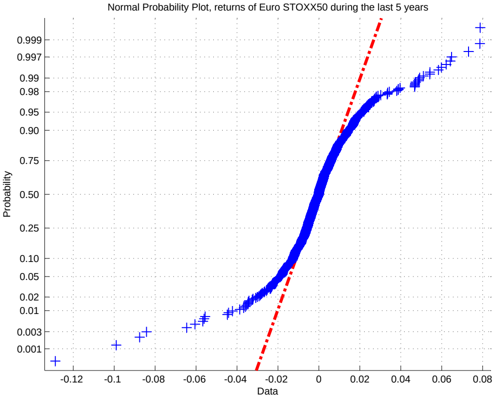

- Lévy processes are able to capture these fat tails!

Let $(\Omega, \mathcal{F}, P)$ be a probability space equipped with the filtration $\{\mathcal{F}_t\}_{t\geq 0}$, where $\mathcal{F}_t$ is the σ-algebra generated by one dimensional Lévy process. We assume that $\mathcal{F}_0$ is augmented of all negligible sets.

Definition

A càdlàg, real valued process $L(t), t \geq 0$ is a 1-dimensional Lévy process with $L(0) = 0$ P-a.s. if the following properties hold

(i) $L(t), t \geq 0$ has independent increments, i.e. for all $0 \leq s < t$ the increment $L(t) - L(s)$ is independent of $\mathcal{F}_s$,

(ii) L has stationary increments, i.e. $L(t) - L(s)$ has the same probability law as $L(t-s)$ for all $0 \leq s < t$,

(iii) L is stochastically continuous, i.e. for all $t \geq 0$ and $\epsilon > 0$ $\lim_{s\to t} P(|L(t) - L(s)| > \epsilon) = 0$.

Lévy-Itô Decomposition

$$dL(t) = \sigma(t)dB(t) + \int_{|x|<1} x\tilde{N}(dt, dx) + \int_{|x|\geq 1} xN(dt, dx) \quad (2)$$

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   - Bayesian Statistics
5. Forecasting

At First Glance

- The Black & Scholes or Black-Scholes-Merton model is a mathematical model of a financial market containing derivative investment instruments.
- It gives a theoretical estimate of the price of European type options.
- The formula led to a boom in options trading and legitimized scientifically the activities of the Chicago Board Options Exchange and other options markets around the world.
- Many empirical tests have shown that the Black & Scholes price is "fairly close" to the observed prices.

Insights of Black & Scholes Model

- The Black & Scholes model was first published by Fischer Black and Myron Scholes in their seminal paper, "The Pricing of Options and Corporate Liabilities, published in the Journal of Political Economy (1973) (Black & Scholes PDE was derived).
- The key idea behind the model is to hedge the option by buying and selling the underlying asset (replicate). This type of hedging is called delta hedging.
- Robert C. Merton (1973) was the first to publish a paper on the mathematical understanding of the options pricing model. Merton and Scholes received the 1997 Nobel Memorial Prize in Economic Sciences. Because of his death in 1995, Black was mentioned as a contributor (ineligible).

Roots of Black & Scholes Model

- L. Bachelier (1900) Ph.D. Thesis: “Theory of Speculation”

$$X(t_{n+1}) = X(t_n) + \mu \triangle t + \sqrt{\triangle t} Z_{n+1} \tag{3}$$

- Einstein (1905): connection of diffusion theory (PDE) and molecular physics was made (investigations on the theory of Brownian motion)
- A. Kolmogorov (1933): the theory of probability is defined in terms of measure theory (measurable space, σ-algebra, etc.)
- Stochastic calculus: Itô (1948), R. C. Merton (1969 MIT Ph.D. Thesis)
- P. Samuelson (risk neutral pricing using dynamically adjusted hedging portfolios-shift to the role of the issuer of the option, not buyer (1970, Nobel Prize)
- Black & Scholes model (1997, Nobel Prize in Economic Sciences).

The Financial Market

Given the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$:

- Bank account process (riskless asset) $S_0(t)$,

$$\begin{aligned} dS_0(t) &= S_0(t)r(t)dt, \\ S_0(0) &= 1. \end{aligned}$$

- A risky asset and price process is given by $S_1(t)$,

$$\begin{aligned} dS_1(t) &= S_1(t)[\mu\, dt + \sigma\, dB(t)], \\ S_1(0) &> 0. \end{aligned}$$

Assumptions

- Market is ideal (no constraints on consumption, no transaction costs or taxes, no penalties to short selling).
- Constant riskless interest rate for borrowing or lending.
- No dividend payment during the life of the option.
- The distribution of stock prices is lognormal.
- The variance of the rate of the return on the stock is constant.
- It is possible to borrow money to buy stocks.
- Continuous trading and shares are infinitely divisible

Meta-Theorem

Theorem

Let n denote the number of underlying traded assets in the model excluding the risk free asset, and let d denote the number of random sources. Generically we then have the following relations:

1. *The model is arbitrage free if and only if $n \leq d$.*
2. *The model is complete if and only if $n \geq d$.*
3. *The model is complete and arbitrage free if and only if $n = d$.*

Black & Scholes model is arbitrage free and complete.

Simulation of the Stock Price

$$S(t) = S(u)\exp\left\{(\mu - 1/2\sigma^2)(t-u) + \sigma[W(t) - W(u)]\right\}$$

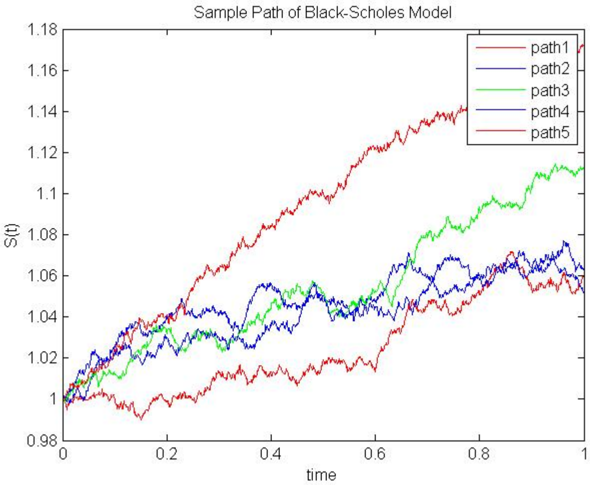
$\Rightarrow$

$$S(t_{n+1}) = S(t_n)\exp\left\{(\mu - 1/2\sigma^2)\triangle t + \sigma\sqrt{\triangle t}\, Z_{n+1}\right\}$$

One can easily show that

$$\mathbb{E}[S(t)] = S(0)e^{\mu t}, \quad Var[S(t)] = S(0)^2 e^{2\mu t}(e^{\sigma^2 t} - 1)$$

Simulation of Stock Price



Risk Neutral Pricing

Theorem

In the Black & Scholes model, any option defined by a non-negative, square integrable (under $\mathbb{Q}$), $\mathcal{F}_T-$measurable random variable h is replicable and the value at time t of any replicating portfolio is given by

$$V_t = E^{\mathbb{Q}}[e^{-\int_t^T r_s \, \mathrm{d}s} h \mid \mathcal{F}_t]. \tag{4}$$

European Call Option:

$$h = f(S_T) = max\,(S_T - K, 0) = (S_T - K)^+ .$$

Asian Option (exotic option):

$$h = \left(\frac{1}{T} \int_0^T S_t dt - K \right)^+ .$$

Analytic Solution of European Call Option

$$C(t, T) = S\,N(d_1) - Ke^{-r(T-t)}N(d_2)$$

V: call premium at time t
S: stock price at time t
T : option exercise time (maturity)
$T - t =: \tau$: time to maturity (T)
K : exercise price
r: risk free interest rate
N : cumulative distribution function

$$d_2 = \frac{ln(\frac{S}{K}) + \left(r - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}, \quad d_1 = d_2 + \sigma\sqrt{\tau}$$

Put-Call Parity:

$$P(t, T) = Ke^{-r(T-t)} - S_t + C(t, T)$$

Black-Scholes Partial Differential Equation

Proposition

Let $V(t, x) \in \mathbb{C}^{1,2}$ *be the price of the European type option at time* t *with the underlying security price* x*. Then* V *satisfies the following partial differential equation for* $(t, x) \in [0, T) \times (0, \infty)$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V}{\partial x^2} + rx\frac{\partial V}{\partial x} - rV = 0,$$

with $V(T, X_T) = h$*.*

Option Price and Implied Volatility

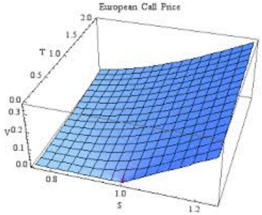


Figure: $K = 1$, $r = 0.04$ and $\sigma = 0.2$

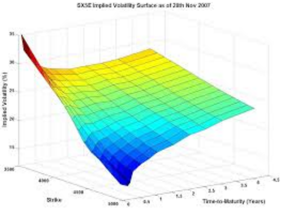


Figure: Eurostoxx 50 index on 28/11/2007

Zero-coupon Bond

The future is certain:

$$P(t, T) = e^{-(T-t)R(t,T)} \tag{5}$$

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$T \to R(t, T)$ What happens if $R(t, T)$ is a stochastic variable?

Zero-coupon Bond

The future is certain:

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$T \rightarrow R(t,T)$ What happens if $R(t,T)$ is a stochastic variable?

- Term structure modeling (time continuous models for $R(t,T)$)(SDEs!)
- Stochastic models for $R(t,T)$ in t and T (T is not fixed)(SPDEs, infinite dimensional modeling problem! $(R(t,t+x))$)

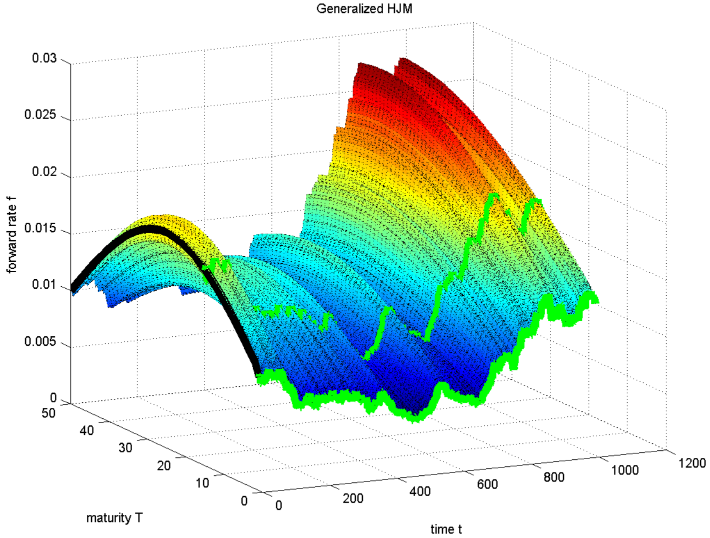
$$P(t,T) = \exp\Big(-\int_t^T f(t,s)ds\Big) \tag{6}$$

The forward rate is the future yield on a bond!

Modeling in the infinite dimensional case

The infinite dimensional modeling of the term structure of interest rates:

1. Modeling of the dynamics of forward rates in *time* and *space* (time to maturity as a space variable)
2. Modeling of the forward rates with infinitely many sources of noise


Generalized HJM
forward rate f
0.03
0.025
0.02
0.015
0.01
0.005
0
maturity T
50
40
30
20
10
0
time t
0
200
400
600
800
1000
1200

Greeks

Name	Symbol	Sensitivity
Delta	Δ	$\frac{\partial V_0}{\partial S_0}$
Gamma	Γ	$\frac{\partial^2 V_0}{\partial S_0^2}$
Rho	ρ	$\frac{\partial V_0}{\partial r}$
Theta	Θ	$\frac{\partial V_0}{\partial t}$
Vega	ν	$\frac{\partial V_0}{\partial \sigma}$

Interesting observations:
The Delta of the put option is always smaller than the Delta of buy option. (same exercise price)
The Gamma and Vega of the call and put option are same.

Quantitative Risk Management

Balance Sheet of a Bank

- Asset:
  - Cash
  - Securities: Bonds, stocks, derivatives
  - Other assets
- Liabilities
  - Customer Deposits
  - Bonds issued: senior bond issues, short-term borrowing

The trading book contains assets that are available to trade. (It is supposed to contain assets that are easy to trade-highly liquid)
The Basel rules allow banks to use internal Value-at-Risk (VaR) models to measure market risk in the trading book.

Trading Book Loss

Definition

The value of the trading book is given by

$$V_t = f(t, Z_t),$$

where f is the portfolio mapping which is assumed to be known and $Z_t = (Z_{t,1}, Z_{t,2}, \ldots, Z_{t,n})$ is the vector of risk factors at time t, where each component is observable.

Examples: equity prices, exchange rates, interest rates for different maturities.

Internal Value-at-Risk (VaR)

Definition

Define $X_{t+1} = Z_{t+1} - Z_t$ as the risk factor change. Assuming positions are held over the period $[t, t+1]$, the trading book loss is described by a random variable

$$\begin{aligned} L_{t+1} = -(V_{t+1} - V_t) &= -\left(f\left(t+1, Z_{t+1}\right)\right) - f\left(t, Z_t\right)) \\ &= -\left(f\left(t+1, Z_t + X_{t+1}\right)\right) - f\left(t, Z_t\right)) \\ &= l_{[t]}\left(X_{t+1}\right) \end{aligned}$$

Aim: Estimate Conditional Loss Distribution:

$$F_{L_{t+1}|\mathcal{F}_t}(x) = \mathbb{P}\left(l_{[t]}(X_{t+1}) \leq x \mid \mathcal{F}_t\right)$$

Delta Approximation of Loss

Risk managers at the banks usually approximate the loss operator by a linear or quadratic function. (first order sensitivities $\rightarrow$ linear, second order sensitivities $\rightarrow$ quadratic

Example

European Call Option valued by Black & Scholes Model Risk Factors $\rightarrow$ (log) price (x_1), interest rate (x_2), volatility (x_3)
$V_t = C(t, S_t; r_t, \sigma_t)$
Linear Loss Operator

$$l_{[t]}(x_1, x_2, x_3) = -(\Theta + \triangle S_t x_1 + \rho x_2 + \nu \sigma_t x_3)$$

Exercise: First, calculate the Greeks in BS model and find the trading loss.

Further Studies

- Financial Derivative Pricing (different models for securities, different numerical techniques for option pricing
- No Model (Pricing under model uncertainty, data driven models)
- Stochastic Control Theory
- Asset-Liability Management
- Credit Risk
- Interest Rate Modeling
- Volatility Modeling
- High Frequency Trading
- Market Microstructure

NEVER FORGET THE MATHEMATICAL FOUNDATIONS!

THANK YOU FOR YOUR ATTENTION !

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